5 RATES OF ENERGY RELEASE RATE AND STABILITY OF MULTIPLE-CRACK SYSTEMS AND 3-D CRACKS

In some areas of fracture mechanics, higher order derivatives of energy release rate due to crack extension are required for prediction of the stability of a single crack, stability of multiple crack systems, and the prediction of fatigue crack growth rate. In the case of multiple crack systems, for example, the variation of energy release rate at one crack tip due to the growth of any other crack must be calculated to determine the strength of the interaction. Another use of the higher order derivatives is for size effect models that relate nominal strength to the structure size. In the universal size effect model proposed by Bazant (1995), the first and second derivatives of energy release rate are needed.

Therefore, an important requirement of some fracture mechanics analyses is to evaluate accurately the energy release rate and its higher order derivatives for a body containing multiple cracks subjected to arbitrary loadings, including crack-face loading, thermal loading and body forces. A virtual crack extension method which provided the direct integral forms of energy release rate and its first derivative for a structure containing one crack was introduced by Lin and Abel (1988). This technique maintains all the advantages of the similar virtual crack extension techniques introduced by deLorenzi (1982, 1985), Haber and Koh (1985), and Barbero and Reddy (1990) and adds the capability of calculating higher order derivatives of energy release rate.

In this section, the generalization of the analytical virtual crack extension method for LEFM presented by Lin and Abel (1988) is presented. Derivations are provided for the following situations: the general case of multiple crack systems, the axisymmetric case, inclusion of crack-face and thermal loading, and evaluation of the second order derivative of energy release rate. The salient feature of this method is that the energy release rate and its higher order derivatives for multiple crack systems are computed in a single analysis. In Section 5.1, the general formulation for the rates of energy release rates for a multiple crack system is presented. In Section 5.2, several 2-D numerical examples with exact solutions or with solutions available in the literature are solved to demonstrate the accuracy of the current method. These examples include: a pressurized crack in an infinite plane as an example of crack-face loading; a center cracked infinite plane subjected to a remote stress, to show the evaluation of the second order derivative of energy release rate; a circular crack subjected to symmetric point loads in an infinite 3-D space, to illustrate an axisymmetric case; and a system of multiple, parallel edge cracks subjected to thermal loading in a semi-infinite plane.

5.1 Formulation

In this section, the analytical expressions for rates of energy release rate for a multiply cracked system under arbitrary loading conditions are derived. For all the developments reported herein, it is assumed that each crack tip will be surrounded by a uniform rosette of standard, quarter-point singular elements, as shown in Figure 55. A discussion of additional mesh perturbation of the nonsingular element layers is also presented.

5.1.1 General formulation for rates of energy release rate for multiply cracked systems

The potential energy Π of a cracked body with multiple cracks is given by

$$\Pi = \frac{1}{2} u^T K u - u^T f \tag{161}$$

where u, K and f are the nodal displacement vector, the structural stiffness matrix and the applied nodal force vector, respectively. The energy release rate at crack tip i can be expressed as,

$$G_i = -\frac{\delta \Pi}{\delta a_i} = -\frac{1}{2} u^T \frac{\delta K}{\delta a_i} u + u^T \frac{\delta f}{\delta a_i} \tag{162}$$

where a_i is the length of crack i, and nonzero contributions to $\delta K / \delta a_i$ and $\delta f / \delta a_i$ occur only over elements adjacent to the crack tip. It is noted that whenever crack-face, thermal and body force loadings are applied, the variations of loading must be taken into account to reflect the local load change on the crack-face or in the crack tip vicinity as a result of virtual crack extension.

The variation of G_i, equation (162), with respect to the growth of any other crack, j, is,

$$\frac{\delta G_i}{\delta a_j} = -u^T \frac{\delta K}{\delta a_i} \frac{\delta u}{\delta a_j} - \frac{1}{2} u^T \frac{\delta^2 K}{\delta a_i \delta a_j} u + \frac{\delta u}{\delta a_j}^T \frac{\delta f}{\delta a_i} + u^T \frac{\delta^2 f}{\delta a_i \delta a_j} \tag{163}$$

We assume that the elements influenced by each crack-tip in a multiply cracked body comprise disjoint sets. Therefore, if $i \neq j$, then the second order variations of stiffness and loading with respect to two different crack extensions a_i and a_j vanish,

$$\frac{\delta^2 K}{\delta a_i \delta a_j} = \frac{\delta^2 f}{\delta a_i \delta a_j} = 0 \tag{164}$$

and, $\delta G_i / \delta a_j$ reduces to,

$$\frac{\delta G_i}{\delta a_j} = -u^T \frac{\delta K}{\delta a_i} \frac{\delta u}{\delta a_j} + \frac{\delta u}{\delta a_j}^T \frac{\delta f}{\delta a_i} \tag{165}$$

The variation of the displacement can be obtained from the variation of the global equilibrium equation $Ku = f$ with respect to a_j,

$$\frac{\delta K}{\delta a_j} u + K \frac{\delta u}{\delta a_j} = \frac{\delta f}{\delta a_j} \quad \text{or} \quad \frac{\delta u}{\delta a_j} = K^{-1}\left(\frac{\delta f}{\delta a_j} - \frac{\delta K}{\delta a_j} u \right) \tag{166}$$

By substituting equation (166) into equation (165), we obtain the final expression for $i \neq j$,

$$\frac{\delta G_i}{\delta a_j} = -u^T \frac{\delta K}{\delta a_i} K^{-1}\left(\frac{\delta f}{\delta a_j} - \frac{\delta K}{\delta a_j} u \right) + \left(\frac{\delta f}{\delta a_j} - \frac{\delta K}{\delta a_j} u \right)^T K^{-1T} \frac{\delta f}{\delta a_i} \tag{167}$$

For the case of $i = j$,

$$\frac{\delta G_i}{\delta a_i} = -u^T \frac{\delta K}{\delta a_i} \frac{\delta u}{\delta a_i} - \frac{1}{2} u^T \frac{\delta^2 K}{\delta a_i^2} u + \frac{\delta u}{\delta a_i}^T \frac{\delta f}{\delta a_i} + u^T \frac{\delta^2 f}{\delta a_i^2} \tag{168}$$

Making similar substitutions, this can be expressed as,

$$\frac{\delta G_i}{\delta a_i} = -u^T \frac{\delta K}{\delta a_i} K^{-1}\left(\frac{\delta f}{\delta a_i} - \frac{\delta K}{\delta a_i} u \right) - \frac{1}{2} u^T \frac{\delta^2 K}{\delta a_i^2} u + \left(\frac{\delta f}{\delta a_i} - \frac{\delta K}{\delta a_i} u \right)^T K^{-1T} \frac{\delta f}{\delta a_i} + u^T \frac{\delta^2 f}{\delta a_i^2} \tag{169}$$

Element stiffness variations $\delta k / \delta \mathrm{a}$ and $\delta^2 k / \delta \mathrm{a}^2$ are assembled to produce the global stiffness variations $\delta K / \delta \mathrm{a}$ and $\delta^2 K / \delta \mathrm{a}^2$. From Lin and Abel (1988), element stiffness variations are given by,

$$\delta k = \int_v \left[\delta B^T DB + B^T D \delta B + Tr(\tilde{\varepsilon}) B^T DB \right] dV \tag{170}$$

$$\delta^2 k = \int_v \left[\delta^2 B^T DB + 2\delta B^T D \delta B + B^T D \delta^2 B + 2\left|\tilde{\varepsilon}\right| B^T DB + 2Tr(\tilde{\varepsilon})\left(\delta B^T DB + B^T D \delta B\right) \right] dV \tag{171}$$

where $\tilde{\varepsilon}$ is the virtual strain-like matrix, B is the strain-nodal displacement matrix, and D the elastic constitutive matrix. $\tilde{\varepsilon}$ is defined as,

$$\tilde{\varepsilon} = J^{-1} \begin{Bmatrix} \partial N / \partial \xi^1 \\ \partial N / \partial \xi^2 \end{Bmatrix} \left\lfloor \Delta_n^1 \quad \Delta_n^2 \right\rfloor = \begin{bmatrix} \tilde{\varepsilon}_{11} & \tilde{\varepsilon}_{12} \\ \tilde{\varepsilon}_{21} & \tilde{\varepsilon}_{22} \end{bmatrix} \tag{172}$$

where Δ 's are the geometry changes of the meshes due to virtual crack extension. Equationa (167) and (169) are seen to be the generalization of equation (2.21) in Lin and Abel (1988) that was developed for a single crack tip.

Once G_i, $\delta G_i / \delta a_j$ and $\delta^2 G_i / \delta a_i{}^2$ are computed, stress intensity factor and its higher order derivatives for a pure Mode I can be expressed by

$$\left(K_I\right)_i = \sqrt{G_i H} \tag{173}$$

$$\frac{\delta (K_I)_i}{\delta a_j} = \frac{1}{2}\frac{\delta G_i}{\delta a_j}\sqrt{\frac{H}{G_i}} \tag{174}$$

$$\frac{\delta^2 (K_I)_i}{\delta a_i^{\ 2}} = \frac{1}{2}\frac{\delta^2 G_i}{\delta a_i^{\ 2}}\sqrt{\frac{H}{G_i}} - \frac{1}{4G_i}\left(\frac{\delta G_i}{\delta a_i}\right)^2\sqrt{\frac{H}{G_i}} \tag{175}$$

where $H = E$ for plane stress condition and $H = E / (1 - \nu^2)$ for plane strain condition. E is Young's modulus and ν is Poisson's ratio.

The application of the proposed method to mixed mode fracture problems is straightforward by making use of Ishikawa's mode decomposition technique with a symmetric mesh in the crack-tip neighborhood (Ishikawa 1979, 1980), or Betti's reciprocal theorem and Yau's mutual energy representation for virtual crack extension method (Stern 1976, Yau 1980). The generalized forms of uncoupled energy release rate, mutual energy release rate and their rates for multiply cracked body are presented in Section 5.1.4.

The described formulation has been implemented in the fracture analysis code, FRANC2D (2002), a workstation-based two-dimensional finite element based code for simulating crack propagation (Wawrzynek 1987a,b). This code performs automatic crack propagation in a variety of materials (Bittencourt 1996).

5.1.2 Additional considerations for crack-face, thermal and body force loading

The global load variations, $\delta f / \delta a$ and $\delta^2 f / \delta a^2$, are produced by assembling the element load variations, $\delta f_e / \delta a$ and $\delta^2 f_e / \delta a^2$,

$$\frac{\delta f}{\delta a} = \sum_e \frac{\delta f_e}{\delta a} \tag{5.16}$$

$$\frac{\delta^2 f}{\delta a^2} = \sum_e \frac{\delta^2 f_e}{\delta a^2} \tag{5.17}$$

The elemental equivalent load variations associated with crack extension for a non-uniform crack-face pressure, p, are given by,

$$\delta f_e = \delta \int_s N^T p \; ds = \int_s \left[N^T \delta p + Tr(\tilde{\varepsilon}) N^T p \right] ds \tag{5.18}$$

$$\delta^2 f_e = \delta^2 \int_s N^T p \, ds = \int_s \left[N^T \delta^2 p + 2Tr(\tilde{\varepsilon}) N^T \delta p + 2\left|\tilde{\varepsilon}\right| N^T p \right] ds \tag{5.19}$$

where N are the shape functions.

If an arbitrary pressure distribution, p, is a function of x, then its variations with respect to crack extension for Mode I are

$$p = p(x) \tag{5.20}$$

$$\frac{\delta p}{\delta a} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial a} = \left[N_k \left(\frac{\partial p}{\partial x} \right)_k \right] \cdot \left[N_k \left(\frac{\partial x}{\partial a} \right)_k \right] \tag{5.21}$$

$$\frac{\delta^2 p}{\delta a^2} = \left[N_k \left(\frac{\partial^2 p}{\partial x^2} \right)_k \right] \cdot \left[N_k \left(\frac{\partial x}{\partial a} \right)_k \right]^2 \tag{5.22}$$

where p_k, $(\partial p/\partial x)_k$ and $(\partial^2 p/\partial x^2)_k$ are the nodal pressure value and its first and second derivatives with respect to the direction x at node k, respectively. The term $(\partial x/\partial a)_k$ represents the change of nodal coordinates at node k in the direction x due to the virtual crack extension, δa, as,

$$\delta x_k = \left(\frac{\partial x}{\partial a}\right)_k \delta a \qquad (5.23)$$

If the mesh perturbation due to virtual crack extension is considered for standard quarter-point crack-tip elements, the variation of nodal coordinates in the direction of x is $\delta x_C = \delta a$ at the crack-tip node and $\delta x_Q = 0.75\delta a$ at quarter point nodes (Figure 5.1a). The term $(\partial x/\partial a)_k$ will have a value of 1 for the degree-of-freedom in the x-direction at the crack-tip node, 0.75 at the quarter point nodes, and zero otherwise. It is noted that the solution is independent of the magnitude of the virtual crack extension, δa. δa is usually taken as unity.

In the same manner, the variations of thermal loading for an isotropic material can be derived as,

$$f_e = \int_v B^T D(\alpha \Delta T)\, dV \qquad (5.24)$$

$$\delta f_e = \int_v \left[\delta B^T D(\alpha\Delta T) + B^T D\delta(\alpha\Delta T) + Tr(\tilde{\varepsilon}) B^T D(\alpha\Delta T)\right] dV \qquad (5.25)$$

$$\delta^2 f_e = \int_v \Big[\delta^2 B^T D(\alpha\Delta T) + 2\delta B^T D\delta(\alpha\Delta T) + B^T D\delta^2(\alpha\Delta T) + 2Tr(\tilde{\varepsilon})\big(\delta B^T D\alpha\Delta T + B^T D\delta(\alpha\Delta T)\big) + 2|\tilde{\varepsilon}|B^T D\delta(\alpha\Delta T)\Big]dV \quad (5.26)$$

where ΔT is the temperature profile and α is the thermal expansion coefficient. If an arbitrary temperature profile, ΔT, is a function of x and y, then its variations with respect to crack extension for Mode I are as follows:

$$\Delta T = \Delta T(x, y) \quad (5.27)$$

$$\frac{\delta\Delta T}{\delta a} = \frac{\partial\Delta T}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial\Delta T}{\partial y}\frac{\partial y}{\partial a} = \frac{\partial\Delta T}{\partial x}\frac{\partial x}{\partial a} = \left[N_k\left(\frac{\partial\Delta T}{\partial x}\right)_k\right]\cdot\left[N_k\left(\frac{\partial x}{\partial a}\right)_k\right] \quad (5.28)$$

$$\frac{\delta^2\Delta T}{\delta a^2} = \left[N_k\left(\frac{\partial^2\Delta T}{\partial x^2}\right)_k\right]\cdot\left[N_k\left(\frac{\partial x}{\partial a}\right)_k\right]^2 \quad (5.29)$$

where ΔT_k, $(\partial\Delta T / \partial x)_k$ and $(\partial^2\Delta T / \partial x^2)_k$ are the nodal temperature change and its first and second derivatives with respect to the direction x at node k, respectively. The derivation for body forces is similar to that of thermal loading and is not shown here.

The derivations for the axisymmetric problem and the second derivative of energy release rate are given later in section 5.1.5 and 5.1.6, respectively.

5.1.3 Perturbation of non-singular elements layers

In this section, the perturbation of additional non-singular, eight-noded quadrilateral element layers is considered. It is expected that the use of a nonsingular element layer in the mesh perturbation will improve the solution accuracy for higher order rates of energy release rate, because nonsingular elements contain additional higher order terms lost to the crack tip elements because of those quarter-point distortions. In the quarter-point crack tip element, the stress field is expressed by the singular term ($\sqrt{r}$) and a constant term, while in a regular eight-noded quadrilateral element, it is represented by the constant term and linear term (r). By perturbing additional layers of nonsingular elements, the high order terms can be included in the computation and the solution accuracy of the high order variations of stress intensity factor is considerably enhanced. When only the first ring of crack tip elements is perturbed as a result of virtual crack extension, δa, the crack tip node is shifted to a new crack tip location by δa, and the quarter points to new quarter points by 0.75 δa (Figure 5.1b). If the second ring of non-singular, eight-noded elements along with the first ring of crack tip elements is perturbed, the nodes in those elements have to be shifted to new locations. We selected the linear perturbation shown in Figure 5.1c. Other perturbation schemes are possible but we have not investigated them. A similar procedure is used for three or more rings of elements. For three different mesh perturbations, three different load variations can be derived.

Consider the constant crack-face pressure, p, for illustration. It is easily shown that only the first order variations of equivalent nodal pressure loading have nonzero values for the virtual crack extension.

$$\delta f_e = \delta \int N^T p \, dl = \delta \int N^T p \, J \, d\xi$$

$$= \int N^T \delta p \, J \, d\xi + \int N^T p \, \delta J \, d\xi = \int N^T p \, \delta J \, d\xi = \int N^T p \, \tilde{\varepsilon} J \, d\xi \qquad (5.30)$$

where $\delta p = 0$ for constant pressure.

$$\delta^2 f_e = \delta \int N^T p \, \delta J \, d\xi = \int N^T p \, \delta^2 J \, d\xi \qquad (5.31)$$

where J and $\tilde{\varepsilon}$ are constants for 1-D. Jacobian variations, δJ and $\delta^2 J$, are as follows:

$$\delta J = \tilde{\varepsilon} J = J^{-1} \left(\sum_i \frac{\partial N_i}{\partial \xi} \Delta_i \right) J = \sum_i \frac{\partial N_i}{\partial \xi} \Delta_i \qquad (5.32)$$

$$\delta^2 J = \delta \tilde{\varepsilon} J + \tilde{\varepsilon} \delta J = -\tilde{\varepsilon}^2 J + \tilde{\varepsilon}^2 J = 0 \qquad (5.33)$$

where N_i are shape functions, and Δ_i are the virtual geometry changes of the meshes as a result of virtual crack extension, δa. Now, when only the first of ring of crack tip elements is perturbed due to an unit virtual crack extension (δa=1.0), Δ_i, δJ and load variation are:

$$\Delta_i = \lfloor 0 \quad 0.75 \quad 1 \rfloor^T \tag{5.34}$$

$$\delta J = \frac{\partial N_i}{\partial \xi}\Delta_i = \lfloor (\xi - 0.5) \quad -2\xi \quad (\xi + 0.5) \rfloor \begin{Bmatrix} 0 \\ 0.75 \\ 1 \end{Bmatrix} = \frac{1}{2}(1 - \xi) \tag{5.35}$$

$$\delta f_e = \int N^T p \delta J \mathrm{d}\xi = \int N^T p \frac{1}{2}(1 - \xi)\mathrm{d}\xi = p \lfloor 1/3 \quad 2/3 \quad 0 \rfloor^T \tag{5.36}$$

When the first and 2nd rings of elements are perturbed, Δ_i, δJ and load variations for each ring are:

$$\Delta_i = \lfloor 0.5 \quad 0.875 \quad 1 \rfloor^T, \qquad \delta J = \frac{1}{4}(1 - \xi) \quad \text{for the first ring} \tag{5.37}$$

$$\Delta_i = \lfloor 0 \quad 0.25 \quad 0.5 \rfloor^T, \qquad \delta J = \frac{1}{4} \quad \text{for the second ring} \tag{5.38}$$

$$\delta f_e = \int N^T p \frac{1}{4}(1 - \xi)\mathrm{d}\xi = p \lfloor 1/6 \quad 1/3 \quad 0 \rfloor^T \quad \text{for the first ring} \tag{5.39}$$

$$\delta f_e = \int N^T p \frac{1}{4}\mathrm{d}\xi = p \lfloor 1/12 \quad 1/3 \quad 1/12 \rfloor^T \quad \text{for the second ring} \tag{5.40}$$

Along the crack face, the total load variation is obtained by adding eqs (5.39) and (5.40)

$$\delta f_e = p \lfloor 1/12 \quad 1/3 \quad 1/4 \quad 1/3 \quad 0 \rfloor^T \tag{5.41}$$

Finally, when the first, 2nd and 3rd rings of elements are perturbed, Δ_i, δJ and load variations for each ring are:

$$\Delta_i = \lfloor 0.75 \quad 0.9375 \quad 1 \rfloor^T, \qquad \delta J = \frac{1}{8}(1-\xi) \quad \text{for the first ring} \tag{5.42}$$

$$\Delta_i = \lfloor 0.5 \quad 0.625 \quad 0.75 \rfloor^T, \qquad \delta J = \frac{1}{8} \quad \text{for the second ring} \tag{5.43}$$

$$\Delta_i = \lfloor 0 \quad 0.25 \quad 0.5 \rfloor^T, \qquad \delta J = \frac{1}{4} \quad \text{for the third ring} \tag{5.44}$$

$$\delta f_e = \int N^T p \frac{1}{8}(1-\xi)\, d\xi = p \lfloor 1/12 \quad 1/6 \quad 0 \rfloor^T \quad \text{for the first ring} \tag{5.45}$$

$$\delta f_e = \int N^T p \frac{1}{8}\, d\xi = p \lfloor 1/24 \quad 1/6 \quad 1/24 \rfloor^T \quad \text{for the second ring} \tag{5.46}$$

$$\delta f_e = \int N^T p \frac{1}{4}\, \mathrm{d}\xi = p \lfloor 1/12 \quad 1/3 \quad 1/12 \rfloor^T \quad \text{for the third ring} \tag{5.47}$$

Summing eqs (5.45), (5.46) and (5.47), the total load variation along the crack face is:

$$\delta f_e = p \lfloor 1/12 \quad 1/3 \quad 1/8 \quad 1/6 \quad 1/8 \quad 1/6 \quad 0 \rfloor^T \tag{5.48}$$

5.1.4 Formulation for mixed mode fracture problem

The mode decomposition technique developed by Ishikawa *et al.* (1979, 1980) can be used to solve the mixed mode fracture problem. Ishikawa *et al.* have shown that the analytical crack-tip field parameters can be decomposed into mode I and mode II components through the use of a symmetric crack-tip mesh, with respect to the local crack-tip coordinate system. Thus, by decomposing nodal displacement vector and load vector, u and f, into mode I and mode II vectors, $u = u_I + u_{II}$ and $f = f_I + f_{II}$, uncoupled energy release rates at crack tip i are

$$G_i = (G_I)_i + (G_{II})_i \tag{5.49}$$

$$(G_I)_i = -\frac{1}{2}(u_I)^T \frac{\delta K}{\delta a_i} u_I + (u_I)^T \frac{\delta f_I}{\delta a_i} \tag{5.50}$$

$$(G_{II})_i = -\frac{1}{2}(u_{II})^T \frac{\delta K}{\delta a_i} u_{II} + (u_{II})^T \frac{\delta f_{II}}{\delta a_i} \tag{5.51}$$

in which subscript I and II represent mode I and II components, respectively.

In the same manner, rates of uncoupled energy release rate are,

$$\frac{\delta G_i}{\delta a_j} = \frac{\delta (G_I)_i}{\delta a_j} + \frac{\delta (G_{II})_i}{\delta a_j} \tag{5.52}$$

For $i \neq j$,

$$\frac{\delta (G_I)_i}{\delta a_j} = -(u_I)^T \frac{\delta K}{\delta a_i} K^{-1} \left(\frac{\delta f_I}{\delta a_j} - \frac{\delta K}{\delta a_j} u_I \right) + \left(\frac{\delta f_I}{\delta a_j} - \frac{\delta K}{\delta a_j} u_I \right)^T K^{-1T} \frac{\delta f_I}{\delta a_i} \tag{5.53}$$

$$\frac{\delta (G_{II})_i}{\delta a_j} = -(u_{II})^T \frac{\delta K}{\delta a_i} K^{-1} \left(\frac{\delta f_{II}}{\delta a_j} - \frac{\delta K}{\delta a_j} u_{II} \right) + \left(\frac{\delta f_{II}}{\delta a_j} - \frac{\delta K}{\delta a_j} u_{II} \right)^T K^{-1T} \frac{\delta f_{II}}{\delta a_i} \tag{5.54}$$

For $i = j$,

$$\frac{\delta(G_I)_i}{\delta a_i} = -(u_I)^T \frac{\delta K}{\delta a_i} K^{-1}\left(\frac{\delta f_I}{\delta a_i} - \frac{\delta K}{\delta a_i} u_I\right) - \frac{1}{2}(u_I)^T \frac{\delta^2 K}{\delta a_i^{\,2}} u_I$$

$$+\left(\frac{\delta f_I}{\delta a_i} - \frac{\delta K}{\delta a_i} u_I\right)^T K^{-1T} \frac{\delta f_I}{\delta a_i} + u^T \frac{\delta^2 f_I}{\delta a_i^{\,2}} \quad (5.55)$$

$$\frac{\delta(G_{II})_i}{\delta a_i} = -(u_{II})^T \frac{\delta K}{\delta a_i} K^{-1}\left(\frac{\delta f_{II}}{\delta a_i} - \frac{\delta K}{\delta a_i} u_{II}\right) - \frac{1}{2}(u_{II})^T \frac{\delta^2 K}{\delta a_i^{\,2}} u_{II}$$

$$+\left(\frac{\delta f_{II}}{\delta a_i} - \frac{\delta K}{\delta a_i} u_{II}\right)^T K^{-1T} \frac{\delta f_{II}}{\delta a_i} + u^T \frac{\delta^2 f_{II}}{\delta a_i^{\,2}} \quad (5.56)$$

Here, it is noted that the mesh in the crack-tip neighborhood must be symmetric.

Another method for solving mixed mode fracture problems is to use Betti's reciprocal theorem and Yau's mutual energy representation for the virtual crack extension method (Stern 1976, Yau 1980). Here, the generalized forms of mutual energy release rate and its derivative for multiply cracked body are presented. The mutual energy release rate at crack tip i is,

$$M_i^{(1,2)} = -\frac{\delta \prod}{\delta a_i} = -\left(u^{(2)}\right)^T \frac{\delta K}{\delta a_i} u^{(1)} + \left(u^{(2)}\right)^T \frac{\delta f^{(1)}}{\delta a_i} + \left(u^{(1)}\right)^T \frac{\delta f^{(2)}}{\delta a_i} \quad (5.57)$$

in which superscripts (1) and (2) represent any two sets of equilibrium states of the elastic body.

The rates of mutual energy release rate are:

For $i \neq j$,

$$\frac{\delta M_i^{(1,2)}}{\delta a_j} = -\left(\frac{\delta u^{(2)}}{\delta a_j}\right)^T \frac{\delta K}{\delta a_i} u^{(1)} - \left(u^{(2)}\right)^T \frac{\delta K}{\delta a_i} \frac{\delta u^{(1)}}{\delta a_j} + \left(\frac{\delta u^{(2)}}{\delta a_j}\right)^T \frac{\delta f^{(1)}}{\delta a_i} + \left(\frac{\delta u^{(1)}}{\delta a_j}\right)^T \frac{\delta f^{(2)}}{\delta a_i} \tag{5.58}$$

For $i = j$,

$$\frac{\delta M_i^{(1,2)}}{\delta a_i} = -\left(\frac{\delta u^{(2)}}{\delta a_i}\right)^T \frac{\delta K}{\delta a_i} u^{(1)} - \left(u^{(2)}\right)^T \frac{\delta^2 K}{\delta a_i^2} u^{(1)} - \left(u^{(2)}\right)^T \frac{\delta K}{\delta a_i} \frac{\delta u^{(1)}}{\delta a_i}$$
$$+ \left(\frac{\delta u^{(2)}}{\delta a_i}\right)^T \frac{\delta f^{(1)}}{\delta a_i} + \left(u^{(2)}\right)^T \frac{\delta^2 f^{(1)}}{\delta a_i^2} + \left(\frac{\delta u^{(1)}}{\delta a_i}\right)^T \frac{\delta f^{(2)}}{\delta a_i} + \left(u^{(1)}\right)^T \frac{\delta^2 f^{(2)}}{\delta a_i^2} \tag{5.59}$$

Displacement variations can be computed from the variation of global equilibrium equation.

$$\frac{\delta u^{(1)}}{\delta a_j} = K^{-1}\left(\frac{\delta f^{(1)}}{\delta a_j} - \frac{\delta K}{\delta a_j} u^{(1)}\right) \tag{5.60}$$

$$\frac{\delta u^{(2)}}{\delta a_j} = K^{-1}\left(\frac{\delta f^{(2)}}{\delta a_j} - \frac{\delta K}{\delta a_j} u^{(2)}\right) \tag{5.61}$$

Now, mutual energy release rate and its rates can be related to stress intensity factor and its rates by,

$$M_i^{(1,2a)} = 2\alpha\left(K_I^{(1)}\right)_i \tag{5.62}$$

$$M_i^{(1,2b)} = 2\alpha\left(K_{II}^{(1)}\right)_i \tag{5.63}$$

$$\frac{\delta M_i^{(1,2a)}}{\delta a_j} = 2\alpha\frac{\delta\left(K_I^{(1)}\right)_i}{\delta a_j} \tag{5.64}$$

$$\frac{\delta M_i^{(1,2b)}}{\delta a_j} = 2\alpha\frac{\delta\left(K_{II}^{(1)}\right)_i}{\delta a_j} \tag{5.65}$$

in which superscript (1) represents equilibrium state of numerical solution, while (2a) and (2b) represent equilibrium states of analytical solutions for pure mode I and pure mode II for the used mesh, respectively.

5.1.5 Formulation for axisymmetric problem

let :

- G_i - the energy release rate at crack i
- a_i - the length of crack i
- k - the element stiffness matrix
- K - the structural stiffness matrix
- f_e - the element load vector
- u - the nodal displacement vector
- p - the crack-face load distribution
- $\tilde{\varepsilon}$ - the virtual strain-like matrix
- N_k - the shape function at node k

r - the radius from z-axis to a point in the cross section

r_{ai} - the radius to the tip of the crack i

B - the strain-nodal displacement matrix

D - the elastic constitutive matrix

J - the jacobian matrix

ΔT - the temperature profile

α - the thermal expansion coefficient

The energy release rate at crack tip i can be expressed as

$$G_i = -\frac{\delta\pi}{\delta a_i r_{ai}} = -\frac{1}{2}u^T \frac{\delta K}{\delta a_i r_{ai}} u + u^T \frac{\delta f}{\delta a_i r_{ai}} \tag{5.66}$$

The variation of G_i of equation (5.66) with respect to the growth of any other crack, j, is

$$\frac{\delta G_i}{\delta a_j r_{aj}} = -u^T \frac{\delta K}{\delta a_i r_{ai}} \frac{\delta u}{\delta a_j r_{aj}} - \frac{1}{2}u^T \frac{\delta^2 K}{\delta a_i \delta a_j r_{ai} r_{aj}} u + \frac{\delta u}{\delta a_j r_{aj}}^T \frac{\delta f}{\delta a_i r_{ai}} + u^T \frac{\delta^2 f}{\delta a_i \delta a_j r_{ai} r_{aj}} \tag{5.67}$$

If $i \neq j$, then the second order variations of stiffness and loading with respect to two different crack extensions a_i and a_j vanish.

$$\frac{\delta^2 K}{\delta a_i \delta a_j r_{ai} r_{aj}} = \frac{\delta^2 f}{\delta a_i \delta a_j r_{ai} r_{aj}} = 0 \tag{5.68}$$

Rewriting equation (5.67), $\delta G_i / \delta a_j r_{aj}$ is

$$\frac{\delta G_i}{\delta a_j r_{aj}} = -u^T \frac{\delta K}{\delta a_i r_{ai}} \frac{\delta u}{\delta a_j r_{aj}} + \frac{\delta u}{\delta a_j r_{aj}}^T \frac{\delta f}{\delta a_i r_{ai}} \tag{5.69}$$

The variation of the displacement can be obtained from the variation of the global equilibrium equation $Ku = f$ with respect to $a_j r_{aj}$

$$\frac{\delta K}{\delta a_j r_{aj}} u + K \frac{\delta u}{\delta a_j r_{aj}} = \frac{\delta f}{\delta a_j r_{aj}} \quad \text{or} \quad \frac{\delta u}{\delta a_j r_{aj}} = K^{-1}\left(\frac{\delta f}{\delta a_j r_{aj}} - \frac{\delta K}{\delta a_j r_{aj}} u \right) \tag{5.70}$$

By substituting equation (5.70) into equation (5.69), we obtain

$$\frac{\delta G_i}{\delta a_j r_{aj}} = -u^T \frac{\delta K}{\delta a_i r_{ai}} K^{-1}\left(\frac{\delta f}{\delta a_j r_{aj}} - \frac{\delta K}{\delta a_j r_{aj}} u \right) + \left(\frac{\delta f}{\delta a_j r_{aj}} - \frac{\delta K}{\delta a_j r_{aj}} u \right)^T K^{-1T} \frac{\delta f}{\delta a_i r_{ai}} \tag{5.71}$$

For the case of $i = j$,

$$\frac{\delta G_i}{\delta a_i r_{ai}} = -u^T \frac{\delta K}{\delta a_i r_{ai}} \frac{\delta u}{\delta a_i r_{ai}} - \frac{1}{2} u^T \left(\frac{\delta^2 K}{\delta a_i^2 r_{ai}^2} - \frac{\delta K}{\delta a_i r_{ai}^3} \right) u$$

$$+ \frac{\delta u}{\delta a_i r_{ai}}^T \frac{\delta f}{\delta a_i r_{ai}} + u^T \left(\frac{\delta^2 f}{\delta a_i^2 r_{ai}^2} - \frac{\delta f}{\delta a_i r_{ai}^3} \right) \tag{5.72}$$

where the overall stiffness variations $\delta K / \delta a_i$ and $\delta^2 K / \delta a_i^2$ are produced by assembling element stiffness variations $\delta k / \delta a_i$ and $\delta^2 k / \delta a_i^2$.

The element stiffness for axisymmetric problem and its variations are

$$k = \int B^T D B r dr dz = \int B^T D B r |J| d\xi d\eta \tag{5.73}$$

$$\delta k = \delta \left[\int B^T D B r |J| d\xi d\eta \right]$$

$$= \int \left[\delta B^T DB + B^T D\delta B + Tr(\tilde{\varepsilon}) B^T DB \right] r |J| d\xi d\eta + \int B^T DB \delta r |J| d\xi d\eta \quad (5.74)$$

$$\delta^2 k = \int \left[\delta^2 B^T DB + 2\delta B^T D\delta B + B^T D\delta^2 B \right] r |J| d\xi d\eta$$

$$+ \int \left[2|\tilde{\varepsilon}| B^T DB + 2\,Tr(\tilde{\varepsilon}) \left(\delta B^T DB + B^T D\delta B \right) \right] r |J| d\xi d\eta$$

$$+2 \int \left[\delta B^T DB + B^T D\delta B + Tr(\tilde{\varepsilon}) B^T DB \right] \delta r |J| d\xi d\eta \quad (5.75)$$

where δr is the change of nodal coordinate as a result of virtual crack extension and expressed by

$$\delta r = \frac{\partial r}{\partial a} \delta a = \left[N_k \left(\frac{\partial r}{\partial a} \right)_k \right] \delta a$$

If the first ring of element surrounding the crack tip is used in the mesh perturbation, the term $(\partial r / \partial a)_k$ for Mode-I crack behavior will have a value of one for the degree of freedom in the x-direction at the crack-tip node, 0.75 at the quarter point nodes and zero otherwise. In the case of non-uniform crack-face loading, the element load variations are given by

$$\delta f_e = \delta \int N^T p\, r |J| d\xi = \int \left[N^T \delta p + Tr(\tilde{\varepsilon}) N^T p \right] |J| d\xi + \int N^T p \delta r\, |J| d\xi \quad (5.76)$$

$$\delta^2 f_e = \int \left[N^T \delta^2 p + 2Tr(\tilde{\varepsilon}) N^T \delta p + 2|\tilde{\varepsilon}| N^T p \right] r |J| d\xi$$

$$+2 \int \left[N^T \delta p + N^T p Tr(\tilde{\varepsilon}) \right] \delta r |J| d\xi \quad (5.77)$$

In the same manner, the variations of thermal loading are

$$\delta f_e = \delta \int B^T D(\alpha \Delta T)\, r|J| d\xi d\eta$$

$$= \int \left[\delta B^T D(\alpha\Delta T) + B^T D \delta(\alpha \Delta T) + Tr(\tilde{\varepsilon}) B^T D(\alpha \Delta T)\right] r|J| d\xi d\eta$$

$$+ \int B^T D(\alpha \Delta T)\, \delta r |J| d\xi d\eta \qquad (5.78)$$

$$\delta^2 f_e = \int \left[\delta^2 B^T D(\alpha\Delta T) + 2\,\delta B^T D \delta(\alpha\Delta T) + B^T D \delta^2(\alpha \Delta T)\right] r|J| d\xi d\eta$$

$$+ \int \left[2 Tr(\tilde{\varepsilon})\left(\delta B^T D \alpha \Delta T + B^T D \delta(\alpha \Delta T)\right) + 2\left|\tilde{\varepsilon}\right| B^T D \delta(\alpha\Delta T)\right] r|J| d\xi d\eta$$

$$+ 2\int \left[\delta B^T D(\alpha\Delta T) + B^T D\delta(\alpha\Delta T) + Tr(\tilde{\varepsilon}) B^T D(\alpha\Delta T)\right] \delta r |J| d\xi d\eta \qquad (5.79)$$

5.1.6 *Derivations for the 2nd order derivative of energy release rate*

let :

G_i	-	the energy release rate at crack i
a_i	-	the length of crack i
k	-	the element stiffness matrix
K	-	the structural stiffness matrix
f_e	-	the element load vector
u	-	the nodal displacement vector
p	-	the crack-face load distribution
$\tilde{\varepsilon}$	-	the virtual strain-like matrix
N_k	-	the shape function at node k
B	-	the strain-nodal displacement matrix

D - the elastic constitutive matrix

J - the jacobian matrix

ΔT - the temperature profile

α - the thermal expansion coefficient

The energy release rate at crack tip i can be expressed as

$$G_i = -\frac{1}{2} u^T \frac{\delta K}{\delta a_i} u + u^T \frac{\delta f}{\delta a_i} \tag{5.80}$$

The variation of this with respect to the crack length, a_i, is

$$\frac{\delta G_i}{\delta a_i} = -u^T \frac{\delta K}{\delta a_i}\frac{\delta u}{\delta a_i} - \frac{1}{2} u^T \frac{\delta^2 K}{\delta a_i^{\,2}} u + \frac{\delta u}{\delta a_i}^T \frac{\delta f}{\delta a_i} + u^T \frac{\delta^2 f}{\delta a_i^{\,2}} \tag{5.81}$$

The second variation of this with respect to the crack length, a_i, is

$$\frac{\delta^2 G_i}{\delta a_i^{\,2}} = -\frac{1}{2} u^T \frac{\delta^3 K}{\delta a_i^{\,3}} u - 2u^T \frac{\delta^2 K}{\delta a_i^{\,2}}\frac{\delta u}{\delta a_i} - u^T \frac{\delta K}{\delta a_i}\frac{\delta^2 u}{\delta a_i^{\,2}}$$

$$-\frac{\delta u}{\delta a_i}^T \frac{\delta K}{\delta a_i}\frac{\delta u}{\delta a_i} + \frac{\delta^2 u}{\delta a_i^{\,2}}^T \frac{\delta f}{\delta a_i} + 2\frac{\delta u}{\delta a_i}^T \frac{\delta^2 f}{\delta a_i^{\,2}} + u^T \frac{\delta^3 f}{\delta a_i^{\,3}} \tag{5.82}$$

where the variations of displacements are

$$\frac{\delta u}{\delta a_i} = K^{-1}\left(\frac{\delta f}{\delta a_i} - \frac{\delta K}{\delta a_i} u\right) \quad \text{and} \quad \frac{\delta^2 u}{\delta a_i^{\,2}} = K^{-1}\left(\frac{\delta^2 f}{\delta a_i^{\,2}} - \frac{\delta^2 K}{\delta a_i^{\,2}} u - 2\frac{\delta K}{\delta a_i}\frac{\delta u}{\delta a_i}\right) \tag{5.83}$$

and the element stiffness variations with respect to crack length are

$$\delta k = \int_v \left[\delta B^T D B + B^T D \delta B + Tr(\tilde{\varepsilon}) B^T D B\right] dV \tag{5.84}$$

$$\delta^2 k = \int_v \left[\delta^2 B^T DB + 2\delta B^T D\delta B + B^T D\delta^2 B\right] dV$$

$$+\int_v \left[2\left|\tilde{\varepsilon}\right| B^T DB + 2\,Tr(\tilde{\varepsilon})\left(\delta B^T DB + B^T D\delta B\right)\right] dV \qquad (5.85)$$

$$\delta^3 k = \int_v \left[\delta^3 B^T DB + 3\delta^2 B^T D\delta B + 3\delta B^T D\delta^2 B + B^T D\delta^3 B\right] dV$$

$$+\,3\int_v \left[\delta^2 B^T DB + 2\delta B^T D\delta B + B^T D\delta^2 B\right] Tr(\tilde{\varepsilon})\, dV$$

$$+\int_v \left[2\left|\tilde{\varepsilon}^2\right| B^T DB + 6\left|\tilde{\varepsilon}\right|\left(\delta B^T DB + B^T D\delta B\right) + 2\left|\tilde{\varepsilon}\right| Tr(\tilde{\varepsilon}) B^T DB\right] dV \qquad (5.86)$$

where $\delta^3 B = -6\left(\tilde{\varepsilon}\right)^3 B$ (5.87)

Element load variations for a non-uniform crack-face loading are

$$\delta f_e = \delta \int_s N^T p \; ds = \int_s \left[N^T \delta p + Tr(\tilde{\varepsilon}) N^T p\right] ds \qquad (5.88)$$

$$\delta^2 f = \delta^2 \int_s N^T p \; ds = \int_s \left[N^T \delta^2 p + 2\,Tr(\tilde{\varepsilon}) N^T \delta p + 2\left|\tilde{\varepsilon}\right| N^T p\right] ds \qquad (5.89)$$

$$\delta^3 f_e = \int_s \left[N^T \delta^3 p + 6\left|\tilde{\varepsilon}\right| N^T \delta p + 2\left|\tilde{\varepsilon}^2\right| N^T p + Tr(\tilde{\varepsilon})\left(3N^T \delta^2 p + 2\left|\tilde{\varepsilon}\right| N^T p\right)\right] ds$$

(5.90)

If an arbitrary load distribution, p, is a function of x, then its variations with respect to crack extension for Mode I (x-direction) are as follows:

$$\frac{\delta p}{\delta a} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial a} = \left[N_k \left(\frac{\partial p}{\partial x}\right)_k\right] \cdot \left[N_k \left(\frac{\partial x}{\partial a}\right)_k\right] \qquad (5.91)$$

$$\frac{\delta^2 p}{\delta a^2} = \left[N_k \left(\frac{\partial^2 p}{\partial x^2} \right)_k \right] \cdot \left[N_k \left(\frac{\partial x}{\partial a} \right)_k \right]^2 \qquad (5.92)$$

$$\frac{\delta^3 p}{\delta a^3} = \left[N_k \left(\frac{\partial^3 p}{\partial x^3} \right)_k \right] \cdot \left[N_k \left(\frac{\partial x}{\partial a} \right)_k \right]^3 \qquad (5.93)$$

where p_k, $(\partial p / \partial x)_k$, $(\partial^2 p / \partial x^2)_k$ and $(\partial^3 p / \partial x^3)_k$ are the nodal load value and its first, second and third derivatives with respect to the direction x at node k.

In the same manner the variations of thermal loading can be derived as described below.

$$f_e = \int_v B^T D(\alpha \Delta T)\, dV \qquad (5.94)$$

$$\delta f_e = \int_v \left[\delta B^T D(\alpha \Delta T) + B^T D \delta(\alpha \Delta T) + Tr(\tilde{\varepsilon}) B^T D(\alpha \Delta T) \right] dV \qquad (5.95)$$

$$\begin{aligned} \delta^2 f_e = \int_v & \left[\delta^2 B^T D(\alpha \Delta T) + 2\delta B^T D \delta(\alpha \Delta T) + B^T D \delta^2 (\alpha \Delta T) \right. \\ & \left. + 2Tr(\tilde{\varepsilon})\left(\delta B^T D \alpha \Delta T + B^T D \delta(\alpha \Delta T) \right) + 2|\tilde{\varepsilon}| B^T D \delta(\alpha \Delta T) \right] dV \end{aligned} \qquad (5.96)$$

$$\begin{aligned} \delta^3 f_e = & \int_v \left[\delta^3 B^T D(\alpha \Delta T) + 3\delta^2 B^T D \delta(\alpha \Delta T) + 3\delta B^T D \delta^2 (\alpha \Delta T) + B^T D \delta^3 (\alpha \Delta T) \right] dV \\ & + 3\int_v \left[\delta^2 B^T D(\alpha \Delta T) + 2\delta B^T D \delta(\alpha \Delta T) + B^T D \delta^2 (\alpha \Delta T) \right] Tr(\tilde{\varepsilon}) dV \end{aligned}$$

$$+\int_v \left[2\left|\tilde{\varepsilon}^2\right| B^T D(\alpha\Delta T) + 6\left|\tilde{\varepsilon}\right|\left(\delta B^T D(\alpha\Delta T) + B^T D\delta(\alpha\Delta T)\right) + 2\left|\tilde{\varepsilon}\right| Tr(\tilde{\varepsilon}) B^T D(\alpha\Delta T) \right] dV \quad (5.97)$$

If an arbitrary temperature profile ΔT is a function of x and y, then its variations with respect to crack extension for Mode I are as follows:

$$\Delta T = \Delta T(x, y) \quad (5.98)$$

$$\frac{\delta \Delta T}{\delta a} = \frac{\partial \Delta T}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial \Delta T}{\partial y}\frac{\partial y}{\partial a} = \frac{\partial \Delta T}{\partial x}\frac{\partial x}{\partial a} = \left[N_k \left(\frac{\partial \Delta T}{\partial x}\right)_k \right] \cdot \left[N_k \left(\frac{\partial x}{\partial a}\right)_k \right] \quad (5.99)$$

$$\frac{\delta^2 \Delta T}{\delta a^2} = \left[N_k \left(\frac{\partial^2 \Delta T}{\partial x^2}\right)_k \right] \cdot \left[N_k \left(\frac{\partial x}{\partial a}\right)_k \right]^2 \quad (5.100)$$

$$\frac{\delta^3 \Delta T}{\delta a^3} = \left[N_k \left(\frac{\partial^3 \Delta T}{\partial x^3}\right)_k \right] \cdot \left[N_k \left(\frac{\partial x}{\partial a}\right)_k \right]^3 \quad (5.101)$$

where ΔT_k, $(\partial \Delta T / \partial x)_k$, $(\partial^2 \Delta T / \partial x^2)_k$ and $(\partial^3 \Delta T / \partial x^3)_k$ are the nodal temperature drop and its first, second and third derivatives with respect to the direction x at node k.

5.2 Numerical Examples

5.2.1 Example 1: a pressurized mode-I crack in an infinite plate

The first numerical example investigates a small, pressurized central crack in a large plate. As shown in Figure 5.2, the initial crack length to width ratio a/W is 0.01 to approximate a central crack in the infinite plate under a plane stress condition. Due to the symmetry in the problem, one-half of the plate was modeled with about 500 linear strain triangular elements including quarter-point elements at the crack-tip (Figure 5.3a). The exact K_I and $\delta K_I / \delta a$ solutions for Mode-I crack growth under uniform crack pressure, p, in an infinite plate can be expressed analytically as,

$$K_I = p\sqrt{\pi a}\,, \qquad \frac{\delta K_I}{\delta a} = \frac{p}{2}\sqrt{\frac{\pi}{a}} \tag{5.102}$$

Table 5.1 compares the computed values of K_I and $\delta K_I / \delta a$ for various crack lengths with the exact solutions. The results are in good agreement with exact solutions, giving maximum errors for K_I and $\delta K_I / \delta a$ of 0.1 % and 2 %, respectively, for this mesh.

5.2.2 Example 2: a center cracked infinite plate subjected to a uniform remote tensile stress

The next example considers the same geometry as that of the first example, but with a different loading condition. The infinite cracked plate is subjected to uniform remote tensile stress, σ_0, instead of crack-face pressure. It should be noted that the $\delta f / \delta a$ and $\delta^2 f / \delta a^2$ terms are null for virtual crack extension under this loading. This model is analyzed to demonstrate the capability of the proposed method for evaluating the second derivative of energy release rate. The exact K_I, $\delta K_I / \delta a$ and $\delta K_I^2 / \delta a^2$ solutions for a Mode-I crack under uniform remote tensile stress, σ_0, in an infinite plate can be expressed analytically as,

$$K_I = \sigma_0 \sqrt{\pi a}\,, \qquad \frac{\delta K_I}{\delta a} = \frac{\sigma_0}{2}\sqrt{\frac{\pi}{a}}\,, \qquad \frac{\delta^2 K_I}{\delta a^2} = -\frac{\sigma_0}{4a}\sqrt{\frac{\pi}{a}} \qquad (5.103)$$

Tables 5.2 and 5.3 show that the best computed solutions differ from the exact by about 0.1 % for K_I, 2 % for $\delta K_I / \delta a$, and 5 - 10 % for $\delta K_I^2 / \delta a^2$, respectively. Table 5.3 shows that the solution accuracy for $\delta K_I^2 / \delta a^2$ is affected considerably by the number of rings of elements surrounding the crack tip that are involved in the mesh perturbation due to the virtual crack extension. When additional rings of surrounding elements from the crack tip are used in the perturbation, increasingly more accurate solutions for $\delta K_I^2 / \delta a^2$ are obtained. The effect of crack tip element size on solution accuracy was also investigated. For example 2 the crack tip element size was decreased by repeatedly dividing the crack-tip element at the mid-point to create a new, regular 8-noded element, and a singular element. As the crack tip element size gets small, the radii of the second and third rings are also decreased. In Tables 5.4-5.6, it is shown that as the crack tip element size is decreased, the solutions for $\delta K_I / \delta a$ and $\delta K_I^2 / \delta a^2$ deteriorate while the accuracy of the computed values of K_I is retained. This solution deterioration can be reduced by perturbing additional rings of nonsingular elements surrounding the first ring of crack tip elements. Hence, it is recommended based on these results that at least one ring of nonsingular elements for $\delta K_I / \delta a$ and two rings for $\delta K_I^2 / \delta a^2$ should be used in the mesh perturbation along with the first ring of crack tip elements for a more accurate analysis.

However, adding additional rings alone does not guarantee uniform convergence as shown, for example, in the last column of Table 5.6. This data indicates that if the crack tip elements get too small, insufficient information from the singular field, which this element only can reproduce, is available for an accurate solution.

5.2.3 Example 3: A circular crack under two symmetric point loads in an infinite space

The next problem considered is a circular crack subjected to two symmetric point loads in an infinite space. An analytical solution (Bonnet 1994) for energy release rate G is

$$G = \frac{(1-\nu^2)P^2\alpha(\kappa+\alpha^2)}{E(\pi h)^3(1+\alpha^2)^4} \qquad \text{where, } \alpha = \frac{a}{h} \text{ and } \kappa = \frac{2-\nu}{1-\nu} \tag{5.104}$$

The crack is located in the xy plane and is loaded by two symmetric point forces, $\pm P$, applied at points $(0,0,\pm h)$ as shown in Figure 5.4. It is interesting to note that the parameter $\alpha\ (a/h)$ governs the crack's stability.

$$\frac{dG}{d\alpha} < 0 \ \ (\alpha > \alpha_{\mathrm{m}}), \qquad \frac{dG}{d\alpha} > 0 \ \ (0 < \alpha < \alpha_{\mathrm{m}}), \qquad \frac{dG}{d\alpha} = 0 \ \ (\alpha = \alpha_{\mathrm{m}}) \tag{5.105}$$

$$\text{with} \qquad \alpha_{\mathrm{m}}^{\ 2} = \frac{\sqrt{16\nu^2 - 72\nu + 105} - 2\nu + 9}{2(2-\nu)} \tag{5.106}$$

Here, when $\alpha > \alpha_{\mathrm{m}}$, the crack growth is stable, and unstable otherwise (Bonnet 1994). For a Poisson's ratio, $\nu = 0.3$, the critical value α_m is about 5.276.

The finite element discretization for this problem is based on axisymmetric elements as shown in Figures 5.5a and 5.5b. A boundary element solution due to Bonnet

(Bonnet 1994) is used as another reference solution. The computed values for K_I and $\delta K_I / \delta a$ are presented along with the exact solutions and the boundary element solution in Figures 5.6a and 5.6b. The results agree well with the reference solutions. The present method gives a computed value of 5.297 for the bifurcation point, α_{m}, which differs by only 1 % from the exact solution.

5.2.4 Example 4: a system of interacting parallel equidistant cracks in a semi-infinite plane

The example considered in this section is a system of thermally-induced, equally spaced, parallel edge cracks in a homogeneous, isotropic, elastic, semi-infinite plane, Figure 5.7. This problem has been studied by many researchers (Bazant 1977, 1979a, b; Nemat-Nasser 1978a, b; Sumi 1980). Other problems of this kind include: shrinkage cracks in drying concrete and polymers, surface cracks in aging wood, thermal cracks in nuclear reactor fuel elements, and desiccation cracks in dried-up lake beds and desserts (Nemat-Nasser 1978a, b; Sumi 1980). In this section, major attention is focused on showing the computational capacity of the proposed method for evaluating rates of stress intensity factors of the system of multiple cracks under thermal loading.

First, the problem is briefly reviewed and later numerical results are presented. In this problem, the cracks are initially formed perpendicular to the free surface by cooling at the free surface. As the thermal gradient penetrates into the half-plane due to heat convection or conduction, the cracks start to grow. Quasi-static critical crack growth is assumed. That is, the cracks grow in such a manner that the stress intensity factor K_I remains at the critical value K_{IC}, but slowly enough that inertial effects are negligible. Here, the loading parameter is the cooling penetration depth, D, in which an appreciable temperature gradient has been formed in the half-plane. When crack spacing is larger than crack length, the cracks interact weakly and grow at an equal rate. As crack length becomes comparable to the spacing, interaction becomes significant. Finally, at the critical state, one crack arrests and the other cracks grow faster with increasing loading. This critical state of crack propagation bifurcation corresponds to vanishing diagonal terms in the matrix $\partial(K_I)_i / \partial a_j$ (Bazant 1977, 1979a; Nemat-Nasser 1978a, b; Sumi 1980). Hence, for the determination of this critical state, rates of stress intensity factors for the system should be accurately calculated. In Bazant (1977, 1979a), the derivative matrix $\partial(K_I)_i / \partial a_j$ has been calculated using a finite difference approximation that requires two complete analyses. This technique is highly sensitive to the length of virtual crack extension Δa. In contrast, the present method calculates the solution for the rates of stress intensity factor analytically with a single analysis and, therefore, more objective solutions are obtained.

In this example, the bifurcation points are computed for various temperature profiles and compared with those of Bazant (1979a). As D increases, the corresponding value of crack length "a" is calculated such that $K_I^{(1)} = K_I^{(2)} = K_{IC}$. The computed a and D values are used to construct the equilibrium path which represents the equilibrium state for the system of cracks during crack growth. For a given a and D, on the equilibrium path, the matrix $\partial(K_I)_i / \partial a_j$ is analytically calculated, from which the bifurcation point can be determined. Figure 5.7 shows a model of an infinite number of parallel edge cracks having a periodic pattern. The stability of this system can be represented by a set of two symmetric independent crack lengths, a_1 and a_2 (Figure 5.8a and 5.8b). Initial spacing is $2h = 1\ m$. The following material properties of typical graphite are used for the calculations: Young's modulus $E = 37{,}600 MN / m^2$; Thermal expansion coefficient $\alpha = 8 \times 10^{-6} / ^\circ C$; Poisson's ratio $\nu = 0.305$; Critical stress intensity factor $K_c = 2.94 MNm^{-3/2}$. Initial surface temperature drop ΔT_0 is taken as $100^\circ C$ and fixed during the crack growth. Figure 5.9 shows five different temperature profiles used in this example. These are the same as those used by Bazant (1979a).

In Figure 5.10, the equilibrium paths and bifurcation points for various temperature profiles are presented. It is shown that the location of bifurcation $a / 2h$ and a / D ratios are strongly influenced by the steepness of the cooling front in the given temperature profile. Longer crack lengths for the same D are found with the steeper cooling front, with no bifurcation found for profile 5. Table 5.7 shows that the computed bifurcation points for various temperature profiles are within about 2 % of those of Bazant (1979a).

5.3 Summary

The analytical expressions are presented for higher order derivatives of energy release rates for a multiply cracked system. This paper provides derivations for the following extensions to the work of Lin and Abel: extension to the general case of multiple crack systems, extension to the axisymmetric case, inclusion of crack-face and thermal loading, and evaluation of the second order derivative of energy release rate. The salient feature of this method is that the energy release rate and its higher order derivatives for multiple crack systems are computed in a single analysis. The present formulation has been implemented in FRANC2D, fracture analysis software developed at the Cornell Fracture Group. It is demonstrated through several 2-D numerical examples that the proposed method gives very accurate results for higher order derivatives of energy release rates for single or multiple cracks. It is also shown that the number of rings of elements surrounding the crack tip that are involved in the mesh perturbation due to the virtual crack extension has an effect on the solution accuracy for higher order derivatives of energy release rate. When more rings of surrounding elements from the crack tip are used in the perturbation, more accurate solutions for the higher order derivatives of energy release rate are obtained. The maximum computed errors were about 0.2 % for energy release rate, 2-3 % for its first derivative and 5-10 % for its second derivative between the simulated solutions and the true infinite medium solutions for the mesh density used in the examples.