4 PREDICTING CRACK TRAJECTORY AND ITS STABILITY

This section addresses the problems of predicting crack trajectory and its stability. In LEFM the T-stress plays an important role in these problems. Here the T-stress is first defined. Then theories based on first order trajectory formulations, in which T-stress is ignored, are presented as baselines for later comparison. First-order trajectory stability formulations that require T-stress calculation are then discussed. The next two sub-sections present models for materials with fracture resistance orthotropy in two and three dimensions. These are logically followed by presentation of theories for crack trajectory with T-stress effects in such materials. These theories accommodate a characteristic length and both mode I and mode II dominance. Finally, recent techniques for accurate calculation of T-stress using the finite element method are presented.

4.1 Linear Elastic Crack Kinking Due to Mixed-Mode Loading Without T-Stress: First Order Kinking Theory

The mixed-mode expressions for the two-dimensional elastic stress field around a crack tip, Figure 33, are given to the first two terms (Williams, 1957) as

$$\sigma_r = \frac{1}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[K_I\left(1+\sin^2\frac{\theta}{2}\right)+\frac{3}{2}K_{II}\sin\theta-2K_{II}\tan\frac{\theta}{2}\right]+\frac{T}{2}(1+\cos 2\theta) \tag{90}$$

$$\sigma_\theta = \frac{1}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[K_I\cos^2\frac{\theta}{2}-\frac{3}{2}K_{II}\sin\theta\right]+\frac{T}{2}(1-\cos 2\theta) \tag{91}$$

$$\sigma_{r\theta} = \frac{1}{2\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[K_I\sin\theta + K_{II}(3\cos\theta - 1)\right] - \frac{T}{2}\sin 2\theta \qquad (92)$$

where K_I and K_{II} are the stress intensity factors associated with mode I and mode II loading as illustrated in Figure 34. The T-stress is the constant component of the stress field, and oriented parallel to the crack tip as shown in Figure 33.

Most of the studies related to crack turning found in the literature focus on determining the kink angle that occurs when a crack is loaded with in-plane asymmetry. The leading stress terms are singular in r, thus dominating crack tip stresses in the elastic solution. Thus, with the tacit assumption that the mechanism by which the crack is directed occurs at or very close to the physical crack tip, the second and higher order terms are often neglected, even though they may be significant at some distance from the crack tip. In these first-order turning theories, the asymmetry is characterized exclusively in terms of the mode mixity, K_{II}/K_I. Second-order kinking theories, which assume a process zone size large enough that T-stress affects the kink angle, are discussed in Section 4.5.

The classical first-order maximum tangential stress ($\sigma_{\theta\theta}$ max) theory, proposed by Erdogan and Sih (1963) for isotropic materials, asserts that the crack will grow in a direction normal to maximum tangential tensile stress. By differentiating the first term in equation 91 with respect to theta and setting it to zero (equivalent to setting $\sigma_{r\theta}=0$) they obtained (shown somewhat rearranged)

$$\frac{K_{II}}{K_I} = \frac{-\sin\Delta\theta_c}{(3\cos\Delta\theta_c - 1)} \tag{93}$$

or,

$$\Delta\theta_c = 2\tan^{-1}\left(\frac{1-\sqrt{1+8(K_{II}/K_I)^2}}{4(K_{II}/K_I)}\right) \tag{94}$$

where $\Delta\theta_c$ is the kink angle. This expression predicts straight crack growth unless $K_{II} \neq 0$, as in asymmetric loading, or in the case of a crack with a perturbed trajectory.

Equation 94 is plotted in Figure 35, along with two other well-known first order linear elastic theories, the maximum energy release rate ($G(\theta)_{max}$) theory proposed by Hussain *et al.* (1974), and the minimum strain energy density theory (S(θ)min) proposed by Sih (1974). For convenience, the data is plotted using the mode mixity parameter

$$M^e = \frac{2}{\pi}\tan^{-1}\left(\frac{K_I}{K_{II}}\right) \tag{95}$$

By assuming other quantities, such as maximum principle stress, maximum hoop strain and/or void growth, numerous first-order kinking criteria have been proposed (e.g. Maiti, 1983; Theocaris, 1989; Shirmohamadi, 1995).

For the most part, all these theories predict quite similar kink angles, particularly as K_{II}<<K_I. Nevertheless, the data of many authors have been correlated with the various theories in an attempt to determine the most accurate. Noteworthy among the empirical studies in this respect is the very meticulous work of Maccagno and Knott (1989, 1991), who, unlike most authors, chose a specimen geometry with near zero T-stress in order to minimize higher-order effects. They also designed their specimens of sufficient thickness to ensure a plain strain state of stress near the crack tip. Their work included testing of plexiglass at room temperature, and various grades of steel at low temperature (–196°C, resulting in transgranular cleavage fracture), and showed that even for moderate amounts of ductility, the initial kink angle was well predicted by the maximum tangential stress theory of Erdogan and Sih. This was true even when the ductility was sufficiently high that they had to resort to an elastic-plastic failure criterion to correlate the fracture initiation loads. However, they cautioned that while this was true of the transgranular cleavage failure mode, it might not be true of other failure mechanisms.

Pook (1971), and Liu (1974) provided crack kinking data for aluminum alloys that also correlates well with the maximum tangential stress theory, at least in the predominately mode I regime.

4.2 Linear Elastic Crack Path Stability in a T-stress Environment: First Order Crack Path Instability Theory

Notwithstanding the foregoing discussion of crack kinking theories, the crack turning problems encountered in many real structural applications are not really crack kinking problems. In an average macroscopic sense, cracks typically initiate normal to the maximum tensile stress, and propagate in a rather smoothly curving fashion as the crack negotiates its way among the structural features of the part. Since the first-order isotropic theories predict crack kinking for non-zero K_{II}, it would appear that the only way for a crack to propagate smoothly is for the crack to follow a path along which $K_{II}=0$. Since all the first-order isotropic theories agree exactly for this condition, the crack path is apparently independent of any first-order theory.

While it is true that at a sufficiently small scale the crack path is not smooth due to material inhomogeneities, microscopic failure phenomena, or fluctuations about a mean loading orientation, these anomalies may be considered random in nature, and may be viewed as perturbations to the crack path. Nevertheless, short of characterizing these perturbations and including them explicitly in a probabilistic analysis, it would appear that the best deterministic estimate of the crack path in an average sense would be the path for which $K_{II}=0$.

The above conclusion seems quite intuitive, and was suggested at least as early as Cotterell and Rice (1980), who further proved that for a crack propagating in pure mode I, the strain energy release rate is locally maximized for a straight crack extension. They started with an approximate kinked crack solution for infinitesimal kinks,

$$\begin{aligned} K_I &= C_{11}k_I + C_{12}k_{II} \\ K_{II} &= C_{21}k_I + C_{22}k_{II} \end{aligned} \tag{96}$$

where k_I and k_{II} are the stress intensities of the lead (unkinked) crack, K_I and K_{II} are the resulting stress intensities at the kink tip, and

$$\begin{aligned} C_{11} &= \tfrac{1}{4}\left[3\cos(\Delta\theta/2) + \cos(3\Delta\theta/2)\right] \\ C_{12} &= -\tfrac{3}{4}\left[\sin(\Delta\theta/2) + \sin(3\Delta\theta/2)\right] \\ C_{21} &= \tfrac{1}{4}\left[\sin(\Delta\theta/2) + \sin(3\Delta\theta/2)\right] \\ C_{22} &= \tfrac{1}{4}\left[\cos(\Delta\theta/2) + 3\cos(3\Delta\theta/2)\right] \end{aligned} \tag{97}$$

that was shown accurate to the second order in $\Delta\theta$. For small angles

$$\begin{aligned} K_I &= C_{11}k_I = k_I\left[1 - \frac{3}{8}(\Delta\theta^2)\right] + O(\Delta\theta^3) \\ K_{II} &= C_{21}k_I = -\frac{\Delta\theta}{2}k_I + O(\Delta\theta^3) \end{aligned} \tag{98}$$

The potential energy release rate is (plane stress)

$$\begin{aligned} G &= \frac{1}{E}\left(K_I{}^2 + K_{II}{}^2\right) \\ &= \frac{1}{E}\left(1 - \frac{\Delta\theta^2}{2}\right)K_I{}^2 + O(\Delta\theta^3) \end{aligned} \tag{99}$$

that is clearly maximized for $\Delta\theta=0$.

Cotterell and Rice then considered the future path of an (initially straight) crack propagating in pure mode I as shown in Figure 36, subject to a small perturbation in k_{II} as the crack reaches the origin of the local coordinate system indicated. They retained the T-stress term in their calculations to observe its influence on the crack path.

Based on a formulation of their own derivation that integrates the tractions due to the lead crack stress field over the developing crack path to obtain the stress intensity factors at the crack tip, they obtained for K_{II},

$$K_{II} = k_{II} + \frac{1}{2}\lambda'(l)k_I - \sqrt{\frac{2}{\pi}}\,T\int_0^l \frac{\lambda'(l)}{\sqrt{l-x}}\,dx \qquad (100)$$

accurate to the first order in λ', the slope of the extending crack. Cotterell and Rice gave evidence that this solution was accurate to within about five percent up until the slope of the extending crack exceeds 15 degrees. Setting $K_{II}=0$ at the developing crack tip,

$$\theta_o = \lambda'(l) - \frac{\beta}{\sqrt{\pi}}\int_0^l \left[\frac{\lambda'(l)}{\sqrt{l-x}}\right]dx \qquad (101)$$

where

$$\begin{aligned} \theta_o &= -2\frac{k_{II}}{k_I} \\ \beta &= 2\sqrt{2}\frac{T}{k_I} \end{aligned} \qquad (102)$$

Note that the expression for the small perturbation angle, θ_o, has been defined in such a way as to be in agreement with equation 94 as K_{II} becomes small compared to K_I.

Solving equation 101 for $\lambda(x)$ using the method of Laplace transforms, Cotterell and Rice obtained

$$\lambda(x) = \frac{\theta_o}{\beta^2}\left[\exp\left(\beta^2 x\right)\operatorname{erfc}\left(-\beta\sqrt{x}\right) - 1 - 2\beta\sqrt{\frac{x}{\pi}}\right] \tag{103}$$

that is plotted in normalized format in Figure 37.

The primary conclusion drawn is that if T>0, the crack path diverges, if T<0, the crack path turns back toward a relatively straight trajectory after the initial perturbation. These behaviors are in qualitative agreement with test data, eg. Leevers *et al*. (1982).

The predicted rate of divergence is proportional to the perturbation and the square of the T-stress. A similar analysis was performed by Sumi (1985), who included one additional higher order term in the stress field expansion, and who was able to obtain additional information about whether the crack was approaching a region of greater stability or instability.

First-order linear elastic crack kinking theory has been presented. First-order refers in this sense to the absence of the T-stress or higher order crack tip field parameters in the crack kinking expression. This infers that the theory either assumes a process zone of negligible size, or allows a finite process zone, but excludes or neglects the presence of higher order terms in the in the analysis. All first order theories predict kinking for non-zero K_{II}, thus inferring that the crack path will be smooth only if K_{II}=0.

Crack path instability theory has been presented in the case where the linear elastic crack kinking theory is construed to admit the presence of T-stress, but (tacitly) with vanishing process zone size so that the kinking theory is first order (K_{II}=0). A divergent crack path is predicted in the presence of positive (tensile) T-stress, which is in qualitative agreement with observation. This behavior will be referred to hereafter as the "first-order crack path instability" associated with the T-stress.

4.3 Fracture Resistance Orthotropy in Two Dimensions

In general, materials may exhibit elastic anisotropy as well as anisotropy in fracture resistance. Both forms of anisotropy can have significant effects on crack trajectory. Many materials, such as wrought metal products, are virtually isotropic elastically, but have a preferred direction of (Mode I) crack propagation resulting from the manner in which the material is processed. Often, as for rolled sheet or plate, the processing is of symmetric character, and the two-dimensional relation describing the crack growth resistance as a function of orientation has two axes of symmetry. This special case is referred to hereafter as two-dimensional fracture toughness orthotropy. For convenience, the orientation describing the crack angle in material coordinates is measured from the longitudinal grain direction, which corresponds to the rolling direction for rolled products. The crack growth resistance is maximum for growth across the rolling direction (θ=90°, or L-T) and minimum for growth parallel to the rolling direction (0°, or T-L), eg. Lemant *et al.* (1981). Materials produced by other processes, such as extrusion, and to a lesser extent, forgings, would be expected to exhibit comparable symmetries, at least locally.

One can approximate the orthotropic crack growth resistance as a function of θ of the form (Chen, 1999)

$$K_p(\theta)^n\left(\frac{\cos^2\theta}{K_p(0°)^n}+\frac{\sin^2\theta}{K_p(90°)^n}\right)=1 \tag{104}$$

where n is a constant exponent. For the present study, K_p is taken to represent the stress intensity at which the crack propagates. It has been proposed (Pettit *et al.,* 2000) that K_p is a material-dependent function of the orientation of the crack tip consistent with the regime of crack growth. Thus, for fatigue crack growth, K_p represents the stress intensity at which the crack propagates at a given rate; for stable tearing, K_p represents the fracture toughness.

In the context of a maximum stress theory, Buczek and Herakovich (1985) suggested a fracture resistance orthotropy relation equivalent to setting n=(-1). They deduced the form of the equation by requiring that the toughness function be independent of θ for isotropic materials, and that it possess the desired orthogonal symmetry, collocating to $K_\mathrm{p}(0)$ and $K_\mathrm{p}(90)$ values. Kfouri (1996) used the more familiar form of an ellipse (n=2). Either case produces a nearly identical oblong shape in polar coordinates for fairly small orthotropy ratios, as illustrated in Figure 38. However, for severe values of orthotropy, positive exponents result in an unjustifiably spiked relationship, as illustrated in Figure 39, plotted in a normalized format given below. In the absence of data to show otherwise, the use of n=(-1) is favored, or

$$K_p(\theta) = K_p(0°)\cos^2\theta + K_p(90°)\sin^2\theta \qquad (105)$$

For the two-dimensional problem we define the normalized crack growth resistance as

$$\overline{K}(\theta) \equiv \frac{K_p(\theta)}{K_p(0^o)} \qquad (106)$$

which varies between unity and $\overline{K}_m$, where $\overline{K}_m$ is the fracture resistance orthotropy ratio defined by

$$\overline{K}_m \equiv \frac{K_p(90°)}{K_p(0°)} \qquad (107)$$

One can rewrite equation 104 in normalized form as

$$\overline{K}(\theta) = \left(\cos^2\theta + \overline{K}_m{}^{-n}\sin^2\theta\right)^{-1/n} \qquad (108)$$

or, for n=(-1)

$$\overline{K}(\theta) = \cos^2\theta + \overline{K}_m \sin^2\theta \qquad (109)$$

Unless otherwise specified, $\overline{K}$ and $\overline{K}_m$ will be assumed to apply to mode I dominated crack growth, and could thus be designated $\overline{K}_I$ and $\overline{K}_{Im}$. For convenience, the modal subscripts will be omitted unless clarity requires them.

4.4 Fracture Resistance Orthotropy in Three Dimensions

In a three-dimensional body a crack may be non-planar, and oriented arbitrarily. At any point along the crack front in an orthotropic material, however, one can characterize the local orientation in terms of the tangent plane and the crack front normal vector within that plane, defined relative to the principal axes of the material.

For an orthotropic material such as a rolled or extruded plate, there are three orthogonal planes of symmetry. Within each of these planes there are thus two orthogonal axes of symmetry. This results in six principal fracture toughness values. The material is assumed to be homogeneous, thus the toughness for a given orientation relative to these principal planes is invariant with regard to translation.

Following the convention established for metals (Goode, 1972), the principle values of fracture toughness are written in a two-letter code (i-j) where the first letter refers to the principle axis normal to the crack plane, and the second subscript identifies the principle axis corresponding to the direction of propagation. These designations have already been mentioned. The standard principal axes for rectangular products (plate, extrusion and forging) correspond to the longitudinal grain orientation (L), the long transverse grain orientation (T), and the short transverse grain orientation (S). Thus, a crack growing normal to the width in the rolling direction of a plate corresponds to the T-L orientation. The (Mode I) fracture resistance in this direction we shall designate as K_{TL}. For convenience and generality, we will use numeric subscripts (1,2,3) in place of the metallurgical (L,T,S). The six principal fracture resistances are thus K_{12}, K_{21}, K_{23}, K_{32}, K_{13}, and K_{31}.

What is needed is a function to interpolate the fracture resistance for arbitrary orientations in terms of these principal values. As illustrated in Figure 40, a crack (or a point on an arbitrary crack front) may propagate in an arbitrary direction defined by unit vector

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \tag{110}$$

where **i**, **j**, and **k** are unit vectors corresponding to the principal material axes x_1, x_2, and x_3. Vector **a** lies within a plane tangent to the developing crack surface at the crack front, which plane is uniquely described by its unit normal vector

$$\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k} \tag{111}$$

The crack orientation is uniquely defined by the direction cosines a_i and n_i. Following the work of Buczek and Herakovich, the interpolation function must

1. Be independent of a_i and n_i for an isotropic material
2. Return the principal fracture resistances for cracks in the corresponding principal orientations.

We seek the lowest order function that can achieve this. Presumably such a function must revert to the two-dimensional form of equation 105 (for this development, we shall assume $n = -1$).

The angles (using right hand rule) describing the trace of **a** on the principle planes are given by

$$\tan(\theta_1) = \frac{a_3}{a_2} \qquad \tan(\theta_2) = \frac{a_1}{a_3} \qquad \tan(\theta_3) = \frac{a_2}{a_1} \tag{112}$$

where the angle subscript refers to the axis normal to the principal plane. The fracture resistance of a crack, were it to lie in a principal plane normal to axis x_k and propagate in the direction of the corresponding trace defined in equation 112, can be interpolated in two dimensions in a manner analogous to equation 105,

$$K_k(\theta_k) = K_{ki} \cos^2 \theta_k + K_{kj} \sin^2 \theta_k \tag{113}$$

Further observing the trigonometric identity

$$\begin{aligned} \cos^2\left(\tan^{-1}\frac{b}{c}\right) &= \frac{c^2}{b^2 + c^2} \\ \sin^2\left(\tan^{-1}\frac{b}{c}\right) &= \frac{b^2}{b^2 + c^2} \end{aligned} \tag{114}$$

and the property of direction cosines

$$a_1^2 + a_2^2 + a_3^2 = 1 \tag{115}$$

one can combine equations 113 and 114 to write

$$\begin{aligned} K_1(\mathbf{a}) &= \frac{1}{1-a_1^2}\left(K_{12}a_2^2 + K_{13}a_3^2\right) \\ K_2(\mathbf{a}) &= \frac{1}{1-a_2^2}\left(K_{23}a_3^2 + K_{21}a_1^2\right) \\ K_3(\mathbf{a}) &= \frac{1}{1-a_3^2}\left(K_{31}a_1^2 + K_{32}a_2^2\right) \end{aligned} \tag{116}$$

In essence, these may be considered as the fracture resistance components of **a** in the principal planes, as illustrated in Figure 41. Presumably, they must be summed in some weighted combination based on crack-plane normal **n** to obtain the effective fracture resistance, $K_p(\mathbf{a},\mathbf{n})$. Since the weight factors must sum to unity to satisfy the isotropic case, it seems reasonable to write

$$\begin{aligned} K_p(\mathbf{a},\mathbf{n}) &= K_1 n_1^2 + K_2 n_2^2 + K_3 n_3^2 \\ &= \frac{n_1^2}{1-a_1^2}\left(K_{12}a_2^2 + K_{13}a_3^2\right) + \frac{n_2^2}{1-a_2^2}\left(K_{23}a_3^2 + K_{21}a_1^2\right) + \frac{n_3^2}{1-a_3^2}\left(K_{31}a_1^2 + K_{32}a_2^2\right) \end{aligned} \tag{117}$$

An inspection of equation 117 shows that it satisfies the criteria previously outlined.

4.5 Crack Turning Theories with Process Zone Effects and Fracture Resistance Orthotropy

The purpose of this section is to develop practical approaches to simulate crack turning in two dimensions, including process zone effects and fracture resistance orthotropy. Two methods with regard to crack path simulation beyond that presented in Section 4.1 are available: a second-order linear elastic approach which includes a single empirical process zone parameter (Finnie and Saith, 1973), and a fully elastic-plastic approach which directs the crack based on the crack tip opening displacement (Sutton *et al.,* 2000; James, 1998). In order to be useful in practical problems, both methods have been extended to account for fracture toughness orthotropy, based on the two-dimensional approach of Section 4.3. However, only the former method will be described in this chapter.

Both methods have been implemented into the FRANC2D or FRANC2D/L (2002) adaptive mesh, finite element fracture simulation environment, building on the work of previous researchers (Pettit *et al.,* 2000). Therefore, it is appropriate to begin with a brief discussion of the piecewise linear manner in which cracks are represented in such simulations.

4.5.1 Representation of a curvilinear crack by a series of segments

In Section 4.2, the statement was made that “the crack turning problems encountered in many structural applications are not really crack kinking problems”. From the first-order theoretical perspective, this meant that a “real” smoothly curving crack followed K_{II}=0 regardless of the kinking theory used, suggesting that the kinking theories are virtually irrelevant. With the exception of the initial kink angle when a crack initiates from a mixed-mode load state, and, provided that process zone effects and toughness anisotropy effects are absent, this is largely true.

Nevertheless, when simulating the crack path, it is typically convenient to represent the curved crack as a union of a series of linear segments, thus involving a series of kinks. In the FRANC2D environment, the need to kink reflects a limitation of the quadratic element type used, as well as to the fact that the small amount of Mode II stress intensity detected at each step is used to determine the incremental path of the crack by virtue of some kinking theory. The program remeshes in the region of the crack tip for each step. Ideally, the path so determined should converge to the “true” crack path as the step size and element size is reduced.

For the isotropic case, following the K_{II}= 0 criterion, a method was proposed by Stone and Babuska (1998) to model the crack path as a C^1 continuous (kink free) series of polynomial segments. They implemented their approach in a p-element program using quadratic segments. Two of the three coefficients associated with the quadratic polynomial of each segment were defined to make the path C^1 continuous. The remaining coefficient was iterated to drive K_{II} at the tip of the crack extension to zero within some tolerance.

Stone and Babuska provided theoretical and numerical evidence to substantiate that this method indeed converges to an arbitrarily accurate approximation (limited by the accuracy of the stress intensity solution) of the true crack path so long as the junctions between segments are at least C^1 continuous. Kinks, of course, are only C^0 continuous, and the theory could not prove convergence in this case. Nevertheless, Stone and Babuska performed a highly accurate analysis of a curvilinear crack spanning an arc of about 27 degrees, and compared the path with paths developed by various sequences of quadratic segments, and also by sequences of linear segments using the kinking criterion of Equation 93. As the step size was reduced, the kink angles also reduced, thus approaching a smooth crack path. While the curvilinear method was seen to converge more rapidly (with fewer segments) than the kinked crack approximation, it seems apparent from their results that if the kink angles are below 10 degrees for the propagating crack, the path is probably of sufficient accuracy for most engineering purposes. With adaptive mesh codes, such convergence is easily obtained, as observed by Wawrzynek and Ingraffea (1987), and Knops (1994), among others.

This is not surprising in view of the results presented in Section 4.2. There it was shown that, in many respects, a slightly curved crack (limited to a threshold of 15 degrees arc by Cotterell and Rice) can be represented to the first order in arc angle by a straight line with an infinitesimal kink at the tip, aligned with the tip of the true curved crack. Thus, one might expect that a slightly curved segment would be sufficiently well represented by a straight segment with an infinitesimal kink oriented tangent to the direction of the next segment (determined according to the criterion K_{II}=0). Note that for small K_{II}/K_I, equation 94 predicts turning angles resulting in K_{II}=0 to the first order in $\Delta\theta$, as can be verified by equation 100.

The above discussion is based on first order, isotropic crack kinking theory, but at least provides support to the notion that for other crack kinking theories, an assemblage of straight segments should converge to a correct theoretical crack path if, as the segment length is reduced, the discrete turning angles become small in regions of curvilinear growth. Of course, this restriction does not apply at the first kink of a crack loaded with mixed mode loading, where the physical fracture response is well represented by a kink.

4.5.2 Second-order, linear elastic, isotropic maximum stress kinking theory

The mixed mode expressions for the elastic stress field around a crack tip including the first two terms are

$$\sigma_r = \frac{1}{\sqrt{2\pi r}}\cos\frac{\Delta\theta}{2}\left[K_I\left(1+\sin^2\frac{\Delta\theta}{2}\right)+\frac{3}{2}K_{II}\sin\Delta\theta-2K_{II}\tan\frac{\Delta\theta}{2}\right]+\frac{T}{2}(1+\cos 2\Delta\theta) \tag{118}$$

$$\sigma_\theta = \frac{1}{\sqrt{2\pi r}}\cos\frac{\Delta\theta}{2}\left[K_I\cos^2\frac{\Delta\theta}{2}-\frac{3}{2}K_{II}\sin\Delta\theta\right]+\frac{T}{2}(1-\cos 2\Delta\theta) \tag{119}$$

$$\sigma_{r\theta} = \frac{1}{2\sqrt{2\pi r}}\cos\frac{\Delta\theta}{2}\left[K_I\sin\Delta\theta+K_{II}(3\cos\Delta\theta-1)\right]-\frac{T}{2}\sin 2\Delta\theta \tag{120}$$

The classical first-order maximum tangential stress theory given by equation 94 maximizes only the first (singular) term of the tangential stress. This expression predicts straight crack growth unless $K_{II}\neq 0$, as in asymmetric loading or in the case of a perturbed crack.

Williams and Ewing (1972) proposed that the crack would propagate in the direction corresponding to the location of maximum tangential stress evaluated at a material dependent finite distance, r_c, ahead of the crack tip, and included the second term in the crack tip stress field expansion. Finnie and Saith (1973) corrected the formulation of Williams and Ewing for the angled crack problem, and Kosai *et al.* (1992) later derived a more general formulation of the same second-order theory by forcing the $\Delta\theta$ derivative of equation 119 to zero at $r=r_c$ to obtain the implicit expression

$$\frac{K_{II}}{K_I}=\frac{-2\sin\frac{\Delta\theta_c}{2}}{\left(3\cos\Delta\theta_c-1\right)}\left[\cos\frac{\Delta\theta_c}{2}-\frac{8}{3}\frac{T}{K_I}\sqrt{2\pi r_c}\cos\Delta\theta_c\right] \tag{121}$$

Note that according to this expression, the crack may turn with sufficient T-stress even if $K_{II}=0$. In this case, equation 121 yields $\Delta\theta_c > 0$ only if Finnie and Saith's inequality is satisfied.

$$r_c > r_o = \frac{9}{128\pi}\left(\frac{K_I}{T}\right)^2 \tag{122}$$

where, for *T>0*, r_o represents the distance forward of the crack tip at which the angle of maximum tangential stress becomes non-zero. In Figure 42, equation 121 is plotted in normalized format using the dimensionless parameter (defined to result in a bifurcation value of unity)

$$\overline{T} \equiv \frac{8}{3}\frac{T}{K_I}\sqrt{2\pi r_c} = \frac{T}{|T|}\sqrt{\frac{r_c}{r_o}} \tag{123}$$

Equation 121 can be rewritten in terms of $\overline{T}$

$$\overline{T} = \frac{\sin\Delta\theta_c + \frac{K_{II}}{K_I}\left(3\cos\Delta\theta_c - 1\right)}{2\sin\frac{\Delta\theta_c}{2}\cos\Delta\theta_c} \tag{124}$$

From Figure 42, straight crack growth is predicted only for the case where K_{II}=0, and $r_o > r_c$. As r_o approaches r_c, the predicted path becomes very sensitive to small amounts of K_{II}.

As should be the case, with $T=0$ or $r_c=0$, the maximum tangential stress theory of equation 124 reduces to the first order theory of equation 93. As has been mentioned, equation 93 can be derived either by maximizing σ_θ or by setting $\sigma_{r\theta}=0$.

In the case of finite T-stress, one might likewise consider enforcing $\sigma_{r\theta}=0$ to obtain

$$\bar{T}_{(\sigma_{r\theta}\to 0)} = \frac{4}{3}\left(\frac{\sin\Delta\theta_c + \frac{K_{II}}{K_I}\left(3\cos\Delta\theta_c - 1\right)}{2\sin\frac{\Delta\theta_c}{2}\cos\Delta\theta_c} \right) \tag{125}$$

$$= \frac{4}{3}\bar{T}$$

that indicates that the two criteria predict identical kink angles if one recognizes that the applicable characteristic lengths are related by

$$r_{c(\sigma_{r\theta}\to 0)} = \frac{16}{9} r_c \tag{126}$$

where the unsubscripted r_c is the characteristic length pertaining to the maximum tangential stress theory defined previously. Thus, the definition of the characteristic length may vary with the criterion used. Before formulating the extension to fracture orthotropy, a discussion with regard to the physical basis and determination of the characteristic length is in order.

4.5.3 The characteristic length, r_c, of the second-order maximum tangential stress theory

Irwin (1960), Dugdale (1960) and others gave approximate expressions for the size of plastic zone in front of a crack tip in elastic-plastic materials. Reasoning that r_c is related to some failure process, it seems probable that the characteristic length associated with the maximum tangential stress crack turning theory should be no larger than the inelastic zones identified.

Rice and Johnson (1969) discussed the role of various characteristic lengths associated with microscopic failure mechanisms in elastic plastic materials, including the crack blunting radius and void spacing, in the context of plain strain fracture problems. As mentioned previously, the characteristic length associated with crack turning was proposed by Williams and Ewing. As an estimate of the characteristic length for PMMA (plexiglass), they referenced a previous work by Constable *et al.* (1970) that identified equivalent flaw sizes based on fatigue thresholds in polyvinyl chloride of the order of 0.064 mm (0.0025 inches). Constable *et al.* conjectured that the equivalent flaw effect might be associated with crazing.

Using photoelastic methods to observe path instability of a nominally symmetric specimen, Ramulu and Kobayashi (1983) experimentally determined r_c for PMMA to be 1.27mm (0.05 inches). This was a considerably larger value than those obtained by prior authors, but Theocaris and Andrianopolis (1982) independently obtained similar results. While the independent corroboration of r_c data from these authors would seem encouraging, the characteristic length estimate was more than an order of magnitude larger than the plane stress Irwin plastic zone size for this material. Further, based on the fracture toughness and critical T-stress for crack path instability in PMMA given by Selvarathinam and Goree (1998), Equation (4.33) would yield a value of r_c= 2.54 mm (0.1 inches). While the cohesive, strain-softening nature of fracture in this material could enlarge the process zone somewhat, these values of r_c would seem too large based on the scale of any known failure mechanism in that material.

Because imperfections or perturbations giving rise to small amounts of K_{II} can be found in any real cracked structure, the onset of path instability in nominally symmetric specimens would be expected to occur at an r_o value in excess of r_c. Note from Figure 42 that the predicted sensitivity even to very small amounts of K_{II} is substantial as one nears the bifurcation. The more sensitive the manner in which the onset of path instability is detected, the larger the overestimate of r_c that might be expected.

Also using photoelastic methods to observe the onset of path instability in symmetric specimens, Streit and Finnie (1980) determined r_c for 7075-T651 aluminum plate to be 0.254 mm (0.010 inches). They described r_c as the distance at which void growth or crack initiation will occur, referencing Rice and Johnson and others. Using values of strength and toughness they provided, their value of r_c is about 0.7 times the size of the plane strain Irwin plastic zone radius (plane strain assumed based on their specimen configuration).

Kosai, Kobayashi, and Ramulu (1992) later estimated r_c for 2024-T3 and 7075-T6 sheet to be 1.52 mm (0.06 inches) based on the lengths of micro-crack branches observed along dynamic fracture surfaces of test specimens. This is considerably larger than the value given by Streit and Finnie for 7075-T651 plate, but the method of determination of r_c is completely different than previous methods, and the thickness of the material used would justify a plane stress assumption. In this case, the characteristic length estimate is about a third of the plane stress Irwin plastic zone size for 7075-T6, and more than an order of magnitude less than the plastic zone size of 2024-T3.

Pettit *et al.* (1997) found that severe path instability occurred consistently in 2024-T3 double cantilever beam specimens at values of r_o at least as high as 2.8 mm (0.11 inches) (the specimen with the shortest crack, and lowest T-stress tested). To obtain an estimate of r_c, the turning radius was plotted as a function of r_o, and extrapolated to zero turning radius (a sharp kink), at critical value of r_o = 1.27 mm (0.05 inches). This was subsequently used as a conservative estimate of r_c, though it tended to underestimate crack turning in crack turning simulations. Chen (1999) used an r_c value of 2.29 mm (0.09 inches) to obtain improved correlation with Pettit's results. A larger r_c would have further improved correlation, but there was concern that r_c was growing too large compared to the K-T dominant zone of the specimen. Also, spurious oscillation was observed in the predicted crack path, and increased with higher values of r_c.

The disagreement in the literature with regard to values of r_c for the various materials tested, and the apparent disparity between some of the values derived from test data and the assumed physical significance of r_c begs reconsideration of the significance of this parameter. Nevertheless, from the equivalence of equation 126, one can realize that the characteristic length may not correspond to the actual size of any particular physical damage phenomenon, but that its use in the maximum tangential stress theory is simply a surrogate for something more complex than the theory describes. However, if the theory is even a decent surrogate, it would be expected that the r_c would be at least proportional in size to some phenomenological length scale.

4.5.4 2nd order maximum tangential stress theory for materials with fracture resistance orthotropy (mode I dominated)

Whereas the isotropic crack turning theory maximizes tangential stress, Buczek and Herakovich (1985) suggested that the crack path in anisotropic materials would follow the maximum of the ratio of the tangential stress to the crack growth resistance obtained by

$$\frac{d}{d(\Delta\theta)}\left(\frac{\sigma_\theta(\Delta\theta)}{\overline{K}(\theta+\Delta\theta)}\right)=0 \tag{127}$$

Separating variables, one obtains

$$\frac{1}{\sigma_\theta(\Delta\theta)}\frac{d\sigma_\theta}{d\theta}=\frac{1}{\overline{K}(\theta+\Delta\theta)}\frac{d\overline{K}}{d\theta}\equiv\Psi \tag{128}$$

Ψ can be obtained in terms of $\overline{K}_m$ using equation 108

$$\Psi(\theta+\Delta\theta_c)=\left(\frac{2}{n}\right)\frac{\beta\sin 2(\theta+\Delta\theta_c)}{1+\beta\cos 2(\theta+\Delta\theta_c)} \quad \text{where} \quad \beta=\frac{\overline{K}_m{}^n-1}{\overline{K}_m{}^n+1} \tag{129}$$

To simplify notation, the argument of Ψ will not be shown explicitly unless it differs from that given above or is required for clarity. Based on the discussion of Section 4.3, a value of n=-1 will be used. Defining Ψ_I with reference to the mode I orthotropy ratio $\overline{K}_{Im}$, evaluating the left hand side of equation 128 with use of equation 119, and solving for $\overline{T}$,

$$\overline{T} = \frac{\sin\Delta\theta_c + \frac{K_{II}}{K_I}\left(3\cos\Delta\theta_c - 1\right) - 2\Psi_I\left[\frac{K_{II}}{K_I}\sin\Delta\theta_c - \frac{1}{3}\left(1 + \cos\Delta\theta_c\right)\right]}{\sin\left(\frac{\Delta\theta_c}{2}\right)\left(2\cos\Delta\theta_c - \Psi_I\sin\Delta\theta_c\right)} \tag{130}$$

Note that for $\overline{K}_{Im}$=1, Ψ_I =0, and equation 130 reverts to the isotropic form of Equation 124.

Equation 130 is plotted in Figure 43 for $\overline{K}_{Im}$=1.6 with various crack orientations, illustrating how the orthotropy influences the location and nature of the bifurcation. As would be expected, a crack propagating in the direction of least crack growth resistance requires a higher K_{II} or T-stress to alter its course. Conversely, a self-similar crack propagating along the direction of maximum crack growth resistance may turn in a compressive T-stress environment given sufficient fracture orthotropy.

One must take care when evaluating equation 130 to obtain maxima, and not minima. The minima occur to the right of the bifurcation line. In order to derive an expression for the value of $\bar{T}$ where the bifurcation occurs, we examine the limiting case of equation 130

$$\lim \Delta\theta_c \to 0, \quad \bar{T} = \frac{2}{\Delta\theta_c}\left(\frac{K_{II}}{K_I} + \frac{2}{3}\Psi_I(\theta + \Delta\theta_c)\right) \tag{131}$$

Note that the theory only predicts straight growth where $\bar{T}$ is below the bifurcation value and

$$\frac{K_{II}}{K_I} = \left(\frac{K_{II}}{K_I}\right)_{crit} = -\frac{2}{3}\Psi_I(\theta) = -\frac{2}{3}\Psi_{Io} \tag{132}$$

where Ψ_{Io} is defined as equation 129 evaluated at $\Delta\theta_c = 0$. If K_{II}/K_I exceeds this value, then $\Delta\theta_c < 0$. The bifurcation value of $\bar{T}$ is obtained when we assume that equation 129 is satisfied and continue with the limit, from which we obtain

$$\bar{T}_{crit} = 1 + \frac{4}{3}\left(\Psi_{Io}{}^2 + \frac{4\beta(\beta + \cos 2\theta)}{n(1 + \beta\cos 2\theta)^2}\right) \tag{133}$$

where β is as given in Equation 129.

4.5.5 2nd order maximum shear stress theory with fracture orthotropy (mode II dominated)

The above theory is mode I dominant, inasmuch as σ_θ is analogous to mode I stress intensity. However, it has been observed that under certain conditions the crack propagates in the direction of maximum $\sigma_{r\theta}$. Following a similar development to the maximum tangential stress theory, the second-order orthotropic maximum shear stress theory can be obtained by maximizing via

$$\frac{d}{d(\Delta\theta)}\left(\frac{\sigma_{r\theta}(\Delta\theta)}{\overline{K}_{II}(\theta+\Delta\theta)}\right)=0 \qquad (134)$$

to obtain

$$\overline{T}_{II}=\frac{2\cos\frac{\Delta\theta_c}{2}\left[\frac{K_I}{K_{II}}+4\Psi_{II}\left((1-3\cos\Delta\theta_c)-\frac{K_I}{K_{II}}\sin\Delta\theta_c\right)\right]-2\left(9\sin\frac{3\Delta\theta_c}{2}+\sin\frac{\Delta\theta_c}{2}\right)+6\frac{K_I}{K_{II}}\cos\frac{3\Delta\theta_c}{2}}{3(2\cos2\Delta\theta_c-\Psi_{II}\sin2\Delta\theta_c)} \qquad (135)$$

where Ψ_{II} is evaluated in terms of the mode II fracture resistance orthotropy ratio, and

$$\overline{T}_{II}\equiv\frac{8}{3}\frac{T}{K_{II}}\sqrt{2\pi r_c} \qquad (136)$$

Selected plots of equation 135 are shown in Figures 44 and 45. In this case, there is no bifurcation, but one must be careful to obtain the global maximum or minimum of the greatest absolute value of the ratio in brackets in equation 135. For the maximum stress theory, transition from mode I to mode II dominated fracture may be postulated to occur when

$$\max\left(\frac{\sigma_\theta(\Delta\theta)}{K_I(0)\overline{K}_I(\theta+\Delta\theta)}\right) \le \max\left|\frac{\sigma_{r\theta}(\Delta\theta)}{K_{II}(0)\overline{K}_{II}(\theta+\Delta\theta)}\right| \tag{137}$$

4.5.6 Second-order linear elastic virtual kink theory

Consider a lead crack under plane stress conditions with a plastic zone as shown in Figure 46a. Compared to an elastic crack, the plastic zone results in additional deformation that can be approximated by a virtual elastic crack kink as shown in Figure 46b. For self-similar crack growth, Wells (1961), used the Irwin plastic zone correction as an approximation of the effective elastic kink length to obtain an estimate of the CTOD. While the appropriate choice of length may remain in question, it is not unreasonable to assert that for a given material and loading, there is a unique kink length, b_c, and orientation, $\Delta\theta_c$, which will best simulate the deformation field as one moves away from the crack tip into the elastic region. One might even postulate that a crack kink so defined would provide a reasonable approximation of the future crack trajectory.

For a crack propagating under steady-state conditions, b_c would be expected to assume a constant, material-dependent value analogous to r_c.

The direct implementation of such a criterion is problematic. Nevertheless, as linear elasticity is approached (as for so-called brittle materials, and also approximately for slow fatigue crack growth), the length of the virtual kink necessarily vanishes. In this limiting case, Cotterel and Rice (1980) concluded, based on a maximum energy release rate argument, that the crack propagates in pure mode I, which is equivalent to the criterion $K_{II}=0$. For a finite (virtual) kink in the presence of non-zero T-stress, setting $K_{II}=0$ does not generally result in maximizing strain energy release rate, nor does it maximize K_I. Nevertheless, depending on the materials and loading conditions, cracks are observed experimentally to develop trajectories corresponding either to pure mode I or pure mode II cracking. Based on this evidence, an isotropic theory is proposed based on the concept that the virtual kink representing the process zone of an extending crack will develop in the direction of either pure mode I or pure mode II crack opening.

Isida and Nishino (1990) (see also Kfouri (1996)) give a solution for a crack in an infinite plate with a kink at one end subject to general in-plane loading. The stress intensity factors at the kink tip, (uppercase) K_I and K_{II}, are expressed in terms of lead crack (lowercase) stress intensities and T as

$$\begin{aligned} K_I &= F_I^{(1)} k_I + F_I^{(3)} k_{II} - F_I^{(2)} T\sqrt{\pi a} \\ K_{II} &= F_{II}^{(1)} k_I + F_{II}^{(3)} k_{II} - F_{II}^{(2)} T\sqrt{\pi a} \end{aligned} \tag{138}$$

where a is the crack length, and $F_n^{(i)}$ are functions of the kink angle, Δθ, and the normalized kink length, b/a, given in polynomial form by Isida and Nishino. The crack length parameter can be eliminated by normalizing in the form

$$\begin{aligned}\frac{K_I}{k_I} &= F_I^{(1)} + F_I^{(3)}\frac{k_{II}}{k_I} - \frac{F_I^{(2)}}{\sqrt{b/a}}\frac{3}{8\sqrt{2}}\overline{T}_b \\ \frac{K_{II}}{k_I} &= F_{II}^{(1)} + F_{II}^{(3)}\frac{k_{II}}{k_I} - \frac{F_{II}^{(2)}}{\sqrt{b/a}}\frac{3}{8\sqrt{2}}\overline{T}_b\end{aligned} \tag{139}$$

where $\sqrt{b/a}$ divides cleanly out of functions $F_n^{(2)}$, and $\overline{T}_b$ is defined with $b=b_c$ in a manner similar to equation 124.

$$\overline{T}_b \equiv \frac{8}{3}\frac{T}{K_I}\sqrt{2\pi b_c} \tag{140}$$

For mode I dominated growth, values of the crack propagation angle, θ_c , can be obtained by varying Δθ to enforce $K_{II} = 0$ for various combinations of k_{II}/k_I and $\overline{T}$, as presented in Figure 47. Also shown for comparison is the second-order maximum tangential stress theory of equation 124 with its characteristic length, r_c. The two theories are fairly equivalent (though not identical) if one recognizes that the characteristic lengths differ at the bifurcation by a constant factor,

$$b_c = 2.21\, r_c \tag{141}$$

The corresponding mode II fracture behavior has not been evaluated, but could easily be developed in the same manner. Nevertheless, from a linear elastic perspective, there is no apparent advantage to this theory over the the maximum stress theory, which is easier to implement, and has already been extended to include toughness orthotropy and modal transition.

4.6 Calculation of the T-Stress

T-stress calculations have been performed by various authors. In one of the earliest studies, Larsson and Carlsson (1973) evaluated the T-stress using finite elements. Later, Leevers and Radon (1982) directly imposed the infinite series solution given by Williams (1957) in a variational approach to obtain estimates of K_I and T simultaneously. They gave estimates of the T-stress in the form of the dimensionless parameter

$$B = \frac{T\sqrt{a}}{K_I} \tag{142}$$

Based on the convergence observed, Leevers and Radon estimated the error in the B values they provided for various specimen geometries to be less than three percent. Sham (1991) used second-order weight functions and a work conjugate integral to calculate T-stresses in various specimen configurations. Fett (1997, 1998) introduced a Green's function approach to calculate T-stresses, and analyzed numerous configurations.

A more approximate displacement correlation method was outlined by Al-Ani and Hancock (1991) that is nevertheless easy to implement in plate and shell codes, and has been utilized in various forms by other authors (Pettit *et al.,* 1997; Knops, 1994; Chen. 1999).

Cardew *et al.* (1985) and Kfouri (1986) computed the T-stress using a modified *J*-integral based on unpublished work of Eshelby, and also gave results for selected specimens based on finite element analyses. Another type of path independent integral based on the Betti-Rayleigh reciprocal theorem has also been proposed by Sladek *et al.* (1976) and also by Yuan and Yang (1998). It was shown to be mathematically equivalent to the J-integral method by Chen *et al.* (2000). By implementing the contour integral solution into a high polynomial order (p-version) finite element program, Chen *et al.* obtained T-stresses that were claimed to be numerically exact to at least five significant figures. The numerical accuracy was verified by way of an exact benchmark solution (a crack tip and surrounding region with the exact boundary conditions applied corresponding to arbitrary combinations of K_I, K_{II}, and T) and a theoretical error relationship

$$e_T = T_{FE} - T = \tilde{e}_T \frac{K_I}{\sqrt{r_1}} \tag{143}$$

where e_T is the error in the computed T-stress, r_I is a characteristic dimension of the integration zone, and $\tilde{e}_T$ is a coefficient related to the discretization error in the vicinity of the integration zone.

Note that equation 143 predicts that the error in the computed T-stress is degraded as the size of the integration domain is reduced—a trend common to both integral methods described. Also, the integration must be performed about a straight segment at the crack tip. This means that when modeling a curvilinear crack, the integration radius cannot exceed the increment dimension, and as the step size is reduced, as required for path convergence, the accuracy of the T-stress solution will be simultaneously degraded.

Nevertheless, using the highly accurate solutions for simple geometries provided by Chen *et al.* as benchmarks, together with the error estimation parameter, it was found that much of the error in the contour integral results is of a systematic nature, and can be corrected *a posteriori* for a given rosette geometry. Following a brief review of the contour integral solution based on the Betti-Rayleigh reciprocal theorem (Chen 1999), an error correction scheme will be discussed, and correction parameters will be determined for the element type and rosette geometry used in FRANC2D.

4.6.1 Contour integral methods

The Betti-Rayleigh reciprocal theorem can be written for a two-dimensional body bounded by a closed curve *S* without body forces as

$$\oint_S (T_i^* u_i - T_i u_i^*) dS = 0 \qquad (144)$$

where T_i represents a set of surface tractions with resulting surface displacements u_i , and T_i* and u_i* are an independent set of surface tractions and corresponding surface displacements, referred to as *auxiliary fields*. By evaluating the integral at a crack tip around the closed four-segment path shown in Figure 48, and recognizing that segments C_+ and C_- are traction free, path independence can be shown with regard to the other two segments.

$$\int_C (T_i^* u_i - T_i u_i^*) dC = \int_{C_\varepsilon} (T_i^* u_i - T_i u_i^*) dC_\varepsilon \qquad (145)$$

By substituting $T_i=\sigma_{ij}n_j$ for each field, where n_j are components of the outward normal vector along the corresponding path segment, we obtain

$$\int_C (\sigma_{ij}^* u_i - \sigma_{ij} u_i^*) n_j dC = \int_{C_\varepsilon} (\sigma_{i,j}^* u_i - \sigma_{ij} u_i^*) n_j \, dC_\varepsilon \qquad (146)$$

Defining ε as a characteristic dimension of path C_ε, the right hand side can be evaluated analytically as $\varepsilon \rightarrow 0$ using the two-dimensional crack tip stress and displacement field solution given by Williams (1957)

$$\sigma_{ij} = \sum_{\lambda=-\infty}^{+\infty} A_\lambda r^{\frac{\lambda}{2}} f_{ij}^{\lambda}(\theta) \tag{147}$$

$$u_{ij} = \sum_{\lambda=-\infty}^{+\infty} B_\lambda r^{\frac{\lambda}{2}+1} g_{ij}^{\lambda}(\theta) \tag{148}$$

where $\frac{\lambda}{2}$ is the eigenvalue, and A_λ and B_λ are the corresponding coefficients for each eigenvalue. By choosing auxiliary fields corresponding to

$$\begin{aligned} \sigma_{ij}^{*} &\sim r^{-\frac{\lambda}{2}-2} \\ u_{i}^{*} &\sim r^{-\frac{\lambda}{2}-1} \end{aligned} \tag{149}$$

in equation 146, coefficients of order $\frac{\lambda}{2}$ alone are obtained. T is of order λ=0, corresponding to auxiliary stresses and displacements in local Cartesian coordinates (see Figure 48) of

$$\begin{aligned} \sigma_{xx}^{*} &= \frac{\cos 2\theta + \cos 4\theta}{2\pi r^2} \\ \sigma_{yy}^{*} &= \frac{\cos 2\theta - \cos 4\theta}{2\pi r^2} \\ \sigma_{xy}^{*} &= \frac{\sin 4\theta}{2\pi r^2} \end{aligned} \tag{150}$$

$$u_x^* = -\frac{\kappa\cos\theta + \cos 3\theta}{8\pi r G}$$
$$u_y^* = \frac{\kappa\sin\theta - \sin 3\theta}{8\pi r G} \tag{151}$$

where G is the shear modulus, κ=(3-ν)/(4+ν) for plane stress, and κ=(3-4ν) for plane strain problems. The T-stress is then obtained for $\varepsilon \rightarrow 0$ as

$$T = \frac{8G}{\kappa+1}\int_{C_\varepsilon}(\sigma_{ij}^* u_i - \sigma_{ij} u_i^*) n_j \, dC_\varepsilon \tag{152}$$

By virtue of the path independence of equation 146, an equivalent integral can be performed numerically using stresses and strains from the finite element analysis (superscript FE)

$$T = \frac{8G}{\kappa+1}\int_{C}(\sigma_{ij}^* u_i^{FE} - \sigma_{ij}^{FE} u_i^*) n_j \, dC \tag{153}$$

or the equivalent domain integral (Chen, 1999)

$$T = \frac{8G}{\kappa+1}\iint_{A}(\sigma_{ij}^* u_i^{FE} - \sigma_{ij}^{FE} u_i^*) q,_j \, dA \tag{154}$$

where *A* is a domain surrounding the crack tip bounded by curves Γ_0 and Γ_1, and function *q* is equal to unity on Γ_0 and zero on Γ_1. For the FRANC2D implementation, the domain *A* is the area comprising the outer ring of the crack tip rosette as illustrated in Figure 49. The integration zone radius, r_I, also shown in Figure 49, is twice the internal radius of the domain for the FRANC2D implementation.

4.6.2 *Error correction scheme*

Equation 143 was derived by recognizing that the stress contribution of the singular terms in the stress field will converge far slower than the contribution of the non-singular terms, leaving an error in the coefficients of all terms proportional to the coefficients of the singular terms[1]. The square root term in the denominator was included due to dimensional considerations, consistent with the form of equations 118 through 120. Based on a similar convergence rate argument, terms of higher order than *T* are expected to contribute little error to the computed value of *T*.

Unlike the error estimation expression given in (Chen, 1999; Chen *et al.,* 2000), we have taken the liberty to write equation 143 with no absolute value signs enforcing that the error measure always be positive. This represents an assertion that we now acknowledge—namely, that the sign and magnitude of $\tilde{e}_T$ is (at least on an average basis) a characteristic of the rosette configuration. Thus, if the value of $\tilde{e}_T$ were known for a given rosette configuration, the systematic error in a T-stress value calculated using that

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[1] In spite of this argument, K_{II} was found to have no pollution effect on the T-stress in numerical experiments.

rosette could be estimated (and thus corrected) via equation 143. The influence of scaling the rosette dimension is captured by way of the length parameter r_1.

The veracity of this assertion can be supported by rewriting equation 143 in the form of the relative error

$$e_{T_{rel}} = \frac{e_T}{T} = \tilde{e}_T \frac{K_I}{T\sqrt{r_1}} \tag{155}$$

The assertion that the relative error in T-stress scales with the dimensionless parameter $K_I / T\sqrt{r_1}$ is supported by the observation that geometrically similar finite element models that differ only in scale (which implies that the integration path is likewise scaled), should give numerically identical error fractions in the computed T-stress (or any other local stress measurement). In essence, the rosette may be considered as a finite element model with imposed boundary conditions representing K_I and T. Recognizing the similarity of all K_I and T fields relative to a characteristic length $(K_I/T)^2$, one may therefore conclude that the combination of such a field with a rosette model of fixed geometry and scale relative to the field characteristic length will be similar (and thus have comparable relative error) to all other rosette/field combinations with the same relative scaling ratio.

Because $\tilde{e}_T$ represents the discretization error in the vicinity of the integration zone, it should thus be relatively constant so long as the mesh geometry, or rosette, within the integration zone is geometrically similar for all problems. The mesh geometry outside of the integration zone is of secondary influence, and may change from problem to problem, thus its effect will be treated as a probabilistic source of error. Nevertheless, provided that the external mesh is reasonably proportioned, the error introduced should be relatively small.

Based on some highly accurate T-stress solutions, we now proceed to numerically verify the foregoing assertions, and determine the value of the error parameter, $\tilde{e}_T$, for the rosette configuration of Figure 49.

4.6.3 High accuracy reference solutions

Chen *et al.* (1999, 2000) implemented the Betti reciprocal type integral into a highly accurate p-finite element code, and analyzed various model configurations. First, a square model of a crack tip was constructed as shown in Figure 50, with a numerically exact external traction distribution imposed representing arbitrary combinations of K_I, K_{II}, and T.

With this model, Chen showed that by increasing the polynomial order of the solution, the relative T-stress error could be reduced to about 10^{-6} with the rosette geometry used and an element shape function order, $p = 11$. Extremely accurate stress intensities were also obtained. The T-stress error data is re-plotted in Figure 51 in terms of $\tilde{e}_T$, showing that for the rosette of Figure 50, $\tilde{e}_T$ is characteristically negative for all p values evaluated, and is a logarithmic function of p. It was also verified by varying load (K_I) and r_1 that $\tilde{e}_T$ is constant for a given level of p.

Having established the extremely tight accuracy of the rosette geometry at high values of *p*, Chen then embedded the same rosette geometry within the meshes of various test specimen geometries to obtain solutions estimated to be within five significant figures of accuracy. The specimen geometries are summarized in Figure 52, and the results tabulated in Table 6.

4.6.4 Calibration of the rosette geometry

Using the data of Table 6 as a type of calibration standard, the error parameter, $\tilde{e}_T$, was determined for the FRANC2D rosette configuration of Figure 49. To do this, FRANC2D models were created of the various specimen types shown in Figure 52 using the FRANC2D rosette configuration, and the T-stresses were calculated using the methods of Section 4.6.1. A range of integration radii were included for each specimen configuration to provide more data points (each also representing a unique outer mesh). The error, e_T, in the as-calculated T-stress for each case was then determined by

$$e_T = T_{FE} - T_{ref} \tag{156}$$

where T_{ref} is the reference T-stress value from Table 6 corresponding to the specimen geometry and loading. As plotted in Figure 53, it is seen that the error for the FRANC2D rosettes is characteristically negative (the T-stress is underestimated by the FEM).

For each data point, a value of the error parameter was calculated by

$$\tilde{e}_T = \left(T_{FE} - T_{ref}\right) \Big/ \frac{K_I}{\sqrt{r_1}} \tag{157}$$

As would be expected, the $\tilde{e}_T$ values calculated in this way do not agree precisely, but vary according to some distribution function, and can be characterized in terms of mean and standard deviation values:

$$\tilde{e}_T = \tilde{e}_{T_M} \pm \tilde{e}_{T_{StD}} = \text{-}0.00825 \pm .00255 \tag{158}$$

(Mean) (Std. Deviation)

A corrected estimate, T_{cor}, of the T-stress can then be calculated by solving Equation 143 for T, and employing the mean value of $\tilde{e}_T$

$$T \approx T_{cor} = T_{FE} - \tilde{e}_{T_M} \frac{K_I}{\sqrt{r_1}} \tag{159}$$

The standard deviation of the remaining (random) error in T_{cor} can be estimated as

$$Std.\ Deviation \approx \tilde{e}_{T_{StD}} \frac{K_I}{\sqrt{r_1}} \tag{160}$$

A plot of remaining error in the corrected data, e_{Tcor} is shown with lines denoting 50 and 90 percent confidence levels is given in Figure 54. The average error (50 percent confidence level) of the corrected solution was about one fifth of the original error--a significant improvement in accuracy obtained with negligible additional computation.

4.6.5 Other details of potential significance with regard to T-stress calculation

A few other observations arising in the course of the T-stress computation development effort include the following:

1. The presence of K_{II} was not found to incur any numerical pollution into T.
2. The singular elements at the center of the rosette shown in Figure 50 were intentionally omitted from the integration domain, requiring the use of a two-layer rosette configuration. If included, the singular elements resulted in additional scatter in the T-stress calculations, making error correction less effective. The reason for this is not known, but may have something to do with the Gauss integration algorithm as applied to singular elements. An alternate path, not chosen, would be to integrate throughout a single stage rosette without singular elements, and to correct both T and the stress intensities obtained from the J-integral by correction methods similar to those presented above. In this regard, it was found that the systematic component of relative error in K_I was a constant for a given rosette configuration, and is independent of scale.
3. Since the error in the T-stress depends on K_I, and not T, the relative error in T will of course be large if T is small compared to K_I. This is acceptable for the present

crack turning application, because the influence of T is only significant as it becomes large compared to K_I.

4.7 Summary

This section summarized the principal theories for predicting 2D crack trajectory and its stability in a finite element context. The roles of T-stress and r_c in these theories were highlighted, and methods for accurate computation of T-stress were described. Direct extension of these theories to 3D crack shape prediction is still problematic. The next two Sections provide additional theoretical and implementation support for this difficult problem.