3 EXTRACTING STRESS INTENSITY FACTORS AND ENERGY RELEASE RATES FROM FINITE ELEMENT RESULTS

In Volume 2, Chapter 3, it was shown that, under LEFM assumptions, the stress, strain, and displacement fields in the near crack-tip region are determined by the stress intensity factors (SIF's). Therefore, fundamental to the use of the finite element method for LEFM is the extraction of accurate SIF's from the finite element results. A large number of different techniques for extracting SIF's have been presented in the literature. In this section, four techniques are presented: displacement correlation, virtual crack extension, modified crack closure integral, and the *J*-integral. These were selected for their historical importance, simplicity, or their accuracy.

Techniques for extracting SIF's fall into one of two categories: direct approaches, which correlate the SIF's with FEM results directly, and "energy" approaches, which first compute energy release rates. In general, the energy approaches are more accurate and should be used preferentially. However, the direct approaches have utility and are especially useful as a check on energy approaches because the expressions are simple enough that they are amenable to hand calculations.

3.1 Displacement Correlation Methods

Displacement correlation is one of the simplest and historically one of the first techniques used to extract SIF's from FEM results (Chan 1970). It is a direct approach. In its simplest form, the finite element displacements for one point in the mesh are substituted directly into the analytical expressions for near-tip displacements (Chapter 2, Section 3), after subtracting the displacements of the crack tip. Usually, the point is selected to be a node on the crack face where the displacements will be greatest, and thus the relative error in the displacements is expected to be smallest. The configuration for this simple approach is shown in Figure 11.

The expressions for the SIF's using plane strain assumptions are

$$\begin{aligned} K_I &= \frac{\mu\sqrt{2\pi}(v_b - v_a)}{\sqrt{r}(2-2\nu)} \\ K_{II} &= \frac{\mu\sqrt{2\pi}(u_b - u_a)}{\sqrt{r}(2-2\nu)} \\ K_{III} &= \frac{\mu\sqrt{\pi}(w_b - w_a)}{\sqrt{2r}} \end{aligned} \tag{25}$$

where μ is the shear modulus, ν is Poisson's ratio, r is the distance from the crack tip to the correlation point, and u_i, v_i, w_i are the x, y, and z displacements at point i (see Figure 11). The same expressions can be used for plane stress assumptions if ν is replaced with $\nu = \nu / (1+\nu)$.

Nice features of this technique are its simplicity and inherent separation of the SIF's for the three modes of fracture. Unfortunately, to obtain accurate results using the approach care must be taken in selecting the correlation point, and usually a highly refined mesh in the crack tip region is required. The correlation point needs to be selected so that it is clearly in the zone where the K fields dominate. One approach sometimes used with this technique is to compute SIF's for a series of points approaching the crack tip. A curve is then fit through these results and extrapolated to a SIF value for r equals zero.

The SIF's computed by this approach can be improved if quarter-point crack-tip elements (Section 2, above) are used (Shih 1976, Tracey 1977). In this case the displacements along the crack face for the quarter-point element interpolation are, Figure 12:

$$\begin{aligned} v_{upper} &= v_a + (-3v_a + 4v_b - v_c)\sqrt{\frac{r}{l}} + (2v_a - 4v_b + 2v_c)\frac{r}{l} \\ v_{lower} &= v_a + (-3v_a + 4v_d - v_e)\sqrt{\frac{r}{l}} + (2v_a - 4v_d + 2v_e)\frac{r}{l} \end{aligned} \tag{26}$$

The FEM crack opening displacement (COD) is

$$v_{upper} - v_{lower} = \left[4(v_b - v_d) + v_e - v_c\right]\sqrt{\frac{r}{l}} + \left[4(v_b - v_d) + 2(v_c - v_e)\right]\frac{r}{l} \tag{27}$$

The square root term of the FEM COD can then be substituted into the analytical crack-tip displacement field to yield

$$K_I = \frac{\mu\sqrt{2\pi}}{\sqrt{r}(2 - 2\nu)}\left[4(v_b - v_d) + v_e - v_c\right] \tag{28}$$

A similar expression,

$$K_{II} = \frac{\mu\sqrt{2\pi}}{\sqrt{r}(2 - 2\nu)}\left[4(u_b - u_d) + u_e - u_c\right] , \tag{29}$$

is obtained for mode II if the sliding displacements, *u*, are substituted for the opening displacements. Similar expressions are given by Ingraffea and Manu (1980) for 3-D configurations.

3.2 Virtual Crack Extension Methods

The virtual crack extension method is an energy approach that computes the rate of change in the total potential energy of a system for a small (virtual) extension of the crack. Under LEFM assumptions, this is equal to the energy release rate. This method was first proposed by Parks (1975) and by Hellen (1975).

The total potential energy, Π, of a finite element system (in the absence of body forces) is

$$\Pi = \tfrac{1}{2}\mathbf{u}^{\mathbf{T}}\mathbf{K}\mathbf{u} - \mathbf{u}\mathbf{p} \tag{30}$$

where **u** is the nodal displacement vector, **K** is the stiffness matrix and **p** is the eternal force vector. The energy release rate for a small (virtual) crack extension is

$$G = \frac{\partial \Pi}{\partial a} = \tfrac{1}{2}\mathbf{u}^{\mathbf{T}}\frac{\partial \mathbf{K}}{\partial a}\mathbf{u} - \mathbf{u}^{\mathbf{T}}\frac{\partial \mathbf{p}}{\partial a} + \frac{\partial \mathbf{u}^{\mathbf{T}}}{\partial a}\left[\mathbf{K}\mathbf{u} - \mathbf{p}\right] \tag{31}$$

The finite element procedure makes the bracketed term zero. If one makes the simplifying assumption that the external forces do not change during crack growth, then equation 31 simplifies to

$$G = \frac{1}{2}\mathbf{u}^{\mathbf{T}} \frac{\partial \mathbf{K}}{\partial a} \mathbf{u} \tag{32}$$

Parks used a finite difference approximation for $\partial\mathbf{K}/\partial a$

$$\frac{\partial \mathbf{K}}{\partial a} \approx \frac{\mathbf{K}_{a+\Delta a} - \mathbf{K}_{a}}{\Delta a} \tag{33}$$

where only the element stiffness matrices of the elements affected by the virtual crack extension need be considered. Figure 13 shows two possible virtual crack extensions.

In general, the virtual crack extension approach will be more accurate than the displacement correlation approach for a given finite element mesh. However, as originally proposed, only a total energy release rate is computed. It is not separated for the three modes of fracture. This shortcoming can be rectified by a decomposition of the displacements fields as described in the *J*-integral section below.

Haber (1985) substituted an analytical treatment for $\partial\mathbf{K}/\partial a$, which substantially improves the fidelity of the method. Banks-Sills and Sherman (1992) showed that the resulting technique is mathematically equivalent to the equivalent domain version of the *J*-integral, discussed below. Another related technique is that due to Lin (1988), which is discussed in Section 5, below. This approach is particularly useful if derivatives of the energy release rates are required.

3.3 Modified Crack Closure Integral

The modified crack closure integral (MCCI) technique was originally proposed by Rybicki and Kanninen (1977). They observed that Irwin's crack closure integral (1957) could be used as a computational tool. Irwin's integral,

$$\begin{aligned} G_I &= \lim_{\Delta L \to 0} \frac{1}{2\Delta L} \int_0^{\Delta L} \sigma_{yy}(r = x, \theta = 0)\, u_y(r = \Delta L - x, \theta = \pi)\, dr \\ G_{II} &= \lim_{\Delta L \to 0} \frac{1}{2\Delta L} \int_0^{\Delta L} \tau_{xy}(r = x, \theta = 0)\, u_x(r = \Delta L - x, \theta = \pi)\, dr \end{aligned} \quad , \qquad (34)$$

relates the energy release rate to the crack-tip stress and displacement fields for a small crack increment, Figure 14.

Finite element equations can be used to relate the crack-tip stresses to the internal finite element forces near the crack tip so that that equation 34 can be expressed directly in terms of nodal forces and displacements, the primary FEM variables. Furthermore, the fracture modes can be easily separated.

Rybicki and Kanninen discussed the case where linear displacement finite elements are used. For this case, the expressions for G become very simple. In reference to Figure 15a, one analysis can be performed to compute the internal nodal force at the crack tip, $\mathbf{F}^c$. The crack is then extended and a second analysis is performed, Figure 15b, yielding displacements at nodes *c* and *d* ($\mathbf{u}^c$ and $\mathbf{u}^d$). The nodal force and displacement version of equation 34 then reduces to

$$G_I = \frac{1}{2\Delta L} F_y^c \left(u_y^c - u_y^d \right) \quad \text{and} \quad G_{II} = \frac{1}{2\Delta L} F_x^c \left(u_x^c - u_x^d \right). \qquad (35)$$

They further observed that this procedure can be used with only one FEM analysis. If the crack step is small, then the displacements at nodes *c* and *d* in Figure 15b can be closely approximated by the displacements at nodes a and b in Figure 15a. In this case the expressions for the energy release rates are

$$G_I = \frac{1}{2\Delta L} F_y^c \left(u_y^a - u_y^b \right) \quad \text{and} \quad G_{II} = \frac{1}{2\Delta L} F_x^c \left(u_x^a - u_x^b \right). \tag{36}$$

The stress intensity factors can then be computed from the simple relations:

$$K_I = \sqrt{G_I E} \quad \text{and} \quad K_{II} = \sqrt{G_{II} E} \quad \text{for plane stress} \tag{37a}$$

and

$$K_I = \sqrt{G_I E \big/ (1-\nu^2)} \quad \text{and} \quad K_{II} = \sqrt{G_{II} E \big/ (1-\nu^2)} \quad \text{for plane strain} \tag{37b}$$

The signs of the K_I and K_{II} values must be determined from the crack opening and sliding displacements.

The MCCI procedure has been extended for use with higher order element. Of particular interest is its formulation for quarter-point elements due to Ramamurthy *et. al.* (1986). They expressed the crack-tip displacement and stress fields in terms of second order polynomials that were consistent with the quarter-point behavior. After integration of equation 34, the resulting expressions for the G's are (see Figure 16):

$$G_I = \frac{1}{\Delta L}\left[\left(C_{11}F_y^a + C_{12}F_y^f + C_{13}F_y^g\right)\left(u_y^b - u_y^e\right) + \left(C_{21}F_y^a + C_{22}F_y^f + C_{23}F_y^g\right)\left(u_y^c - u_y^d\right)\right]$$
$$G_{II} = \frac{1}{\Delta L}\left[\left(C_{11}F_x^a + C_{12}F_x^f + C_{13}F_x^g\right)\left(u_x^b - u_x^e\right) + \left(C_{21}F_x^a + C_{22}F_x^f + C_{23}F_x^g\right)\left(u_x^c - u_x^d\right)\right] \quad (38a)$$

with

$$C_{11} = \frac{33\pi}{2} - 52, \quad C_{12} = 17 - \frac{21\pi}{4}, \quad C_{13} = \frac{21\pi}{2} - 32$$
$$C_{21} = 14 - \frac{33\pi}{8}, \quad C_{22} = \frac{21\pi}{6} - \frac{7}{2}, \quad C_{23} = 8 - \frac{21\pi}{8} \quad (38b)$$

Figure 16 shows triangular quarter-point elements, but the same expressions can be used with rectangular quarter-point elements.

Raju([1987) presents formulae for a number of additional element types. The method has been further generalized for arbitrary numerical techniques and field interpolations by Singh *et al.* (1998). In general, for the same mesh the MCCI technique yields SIF's that are more accurate than the displacement correlation approach but less accurate than the *J*-integral approach (discussed next). However, it gives surprisingly accurate results for its simplicity and requires nodal forces and displacements only, which are standard outputs from many finite element programs. Terms in addition to those presented here are required if crack face tractions are present.

3.4 The J-Integral (2-D)

The *J*-integral is a well known nonlinear fracture mechanics parameter (Rice, 1968; Cherepanov, 1967; Budiansky and Rice, 1973). Under linear elastic material assumptions, the *J*-integral, *J*, can be interpreted as being equivalent to the energy release rate, *G*. In its original formulation, it relates the energy release rate of a two-dimensional body to a contour integral. Using a crack coordinate system where the x_1 axis is tangential to the crack and the x_2 axis is perpendicular to the crack the *J*-integral is defined as

$$J = \lim_{\Gamma \to 0} \int_{\Gamma} \left[W n_1 - \sigma_{ij} \frac{\partial u_i}{\partial x_1} n_j \right] d\Gamma \qquad (39)$$

where *W* is the strain energy density, **σ** is the stress tensor, **n** is the unit outward normal to the contour, and **u** is the displacement vector (summation convention used over identical indices), see Figure 17. The contour integral in this simple form can be shown to be path-independent providing there are no body forces inside the integration area, there are no tractions on the crack surface, and the material behavior is elastic (linear or nonlinear). Path independence for cases with body forces or crack-face tractions require additional terms in the integral.

Early use of the *J*-integral with finite elements focused on a direct evaluation of equation 39 along a contour in the finite element mesh. Usually, the contour is selected to pass through element Gauss integration points, where stresses are expected to be most accurately evaluated. Unfortunately, such an implementation rarely exhibits path independence of the integral and *ad hoc* procedures must be adopted to obtain an objective value for *J*.

Li *et. al.* (1985) showed how the contour *J*-integral can be transformed to an equivalent area integral, which is much simpler to implement in a finite element context, and has been shown to be objective with respect to the domain of integration (Banks-Sills 1992). The area form of the integral is

$$\bar{J} = \int_A \left[\sigma_{ij} \frac{\partial u_i}{\partial x_1} - W \delta_{1j} \right] \frac{\partial q_1}{\partial x_j} \, dA \tag{40}$$

where δ is the Kronecker delta and q is a weighting function defined over the domain of integration. Physically, q can be thought of as the displacement field due to a virtual crack extension.

The domain of integration can be defined in two ways. Either an annular region that surrounds the crack tip, Figure 18a, or the inner contour can be contracted all the way to the crack tip, Figure 18b. The later case, where only crack-tip elements are used in the integration, is particularly convenient to implement in a finite element program. These cases are conceptually similar to Figure 13 but no actual physical displacements are imposed.

The q function is defined by prescribing nodal values that are interpolated over elements in the domain using the standard shape functions:

$$q = \sum_i N_i q_i \quad \text{and} \quad \frac{\partial q}{\partial x_j} = \sum_i \frac{\partial N_i}{\partial x_j} q_i \tag{41}$$

The other quantities in equation 40 are easily computed in a finite element context ($W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$).

The q function should have a value of one on the inner contour of the domain, Figure 18a, or the crack tip, Figure 18b, and have a value of zero on the outer contour of the domain. A linear spatial variation is usually assumed between the two contours. For example, if the domain of evolution is the crack-tip elements only, and quarter-point elements are used, then the nodal values for q should be one at the crack-tip node, 0.75 at the quarter-point nodes, and zero at all other element nodes.

If there are tractions on the crack faces, an additional term must be added to the *J*-integral. For crack face tractions t_i this is

$$\bar{J} = \bar{J}_A + \bar{J}_\Gamma = \bar{J}_A + \int_{\Gamma_3+\Gamma_4} t_i \frac{\partial u_i}{\partial x_1} q \, d\Gamma \tag{42}$$

where J_A is given by equation 40.

3.4.1 J-Integral mode separation

The *J*-integral as defined in equation 40 gives the total energy release rate for the crack that is

$$\bar{J} = G = \left(K_I^2 + K_{II}^2\right)/E \quad \text{(plane stress)} \tag{43}$$

For mixed-mode crack growth one would like a technique for separating the SIF's due to the different fracture modes. An effective technique for doing this was introduced by Ishikawa et al. (1979, 1980) and independently by Bui (1983). Bui separated the modes by decomposing the near crack-tip displacement fields into one field that is symmetric with respect to the crack and another field that is anti-symmetric with respect to the crack. That is, consider a coordinate system x_1, x_2, centered at the crack tip, with the crack lying on the negative x_1 axis, Figure 19.

Then the local displacement can be expressed as

$$\mathbf{u} = \mathbf{u}^{I} + \mathbf{u}^{II} = \frac{1}{2}\begin{Bmatrix} u_1 + \bar{u}_1 \\ u_2 - \bar{u}_2 \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} u_1 - \bar{u}_1 \\ u_2 + \bar{u}_2 \end{Bmatrix}, \tag{44}$$

with

$$\bar{\mathbf{u}}(x_1, x_2) = \mathbf{u}(x_1, -x_2). \tag{45}$$

A similar decomposition can be used for the stress field

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{I} + \boldsymbol{\sigma}^{II} = \frac{1}{2}\begin{bmatrix} \sigma_{11} + \bar{\sigma}_{11} & \sigma_{12} - \bar{\sigma}_{12} \\ sym & \sigma_{22} + \bar{\sigma}_{22} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \sigma_{11} - \bar{\sigma}_{11} & \sigma_{12} + \bar{\sigma}_{12} \\ sym & \sigma_{22} - \bar{\sigma}_{22} \end{bmatrix}. \tag{46}$$

The $\mathbf{u}^{I}$ field is symmetric about the crack plane and the $\mathbf{u}^{II}$ field is anti-symmetric about this plane.

The mode separated J-integral values can be computed by evaluating equation 40 using the decomposed displacements and stresses. That is,

$$G_I = \bar{J}_I = \bar{J}\left(\mathbf{u}^I\right) \quad \text{and} \quad G_{II} = \bar{J}_{II} = \bar{J}\left(\mathbf{u}^{II}\right) \tag{47}$$

The SIF's can be computed using equation 37.

This modal decomposition technique is simple to implement if the mesh used in the domain of evaluation is symmetric about the crack plane. However, this is not necessary. Interpolation can be used to find displacements and stresses for non-symmetric meshes (Cervenka 1997).

3.5 The J-Integral (3-D)

In three dimensions a *local* value of the *J*-integral, denoted $J(s)$, at each point s on a crack front is given by

$$J(s) = \lim_{\Gamma \to 0} \int_{\Gamma} \left[W n_1 - \sigma_{ij} \frac{\partial u_i}{\partial x_1} n_j \right] d\Gamma \tag{48}$$

where Γ lies in the plane normal to the crack front at s, and all quantities are expressed in the local orthogonal coordinate system located at s. Unlike the *global* path independence of the 2-D version, equation 39, the 3-D *J*-integral is only path independent in a *local* sense as $\Gamma \to 0$ (Moran, 1987). Again, as in 2-D, direct evaluation of equation 48 within a finite element context is difficult because of the need to define a path, Γ, that passes through integration points. Also, the limiting definition of the contour requires a high degree of mesh refinement at the crack front in order to obtain accurate results.

As with 2-D, a weighting function can be introduced to transform the 3-D *J*-integral into a volume integral (Nikishkov and Atluri, 1987). For the case of an elastic material with small strain assumptions, and no body forces within the contour, the expression for the integral is

$$\bar{J} = \int_V \left[\sigma_{ij} \frac{\partial u_i}{\partial x_1} - W\delta_{kj} \right] \frac{\partial q_j}{\partial x_k} \, dV + \int_{A_3 + A_4} t_i \frac{\partial u_i}{\partial x_1} q \, dA \,. \qquad (49)$$

The domain of integration is illustrated in Figure 20.

The *q*-function should be defined so that it vanishes on surfaces A_1, A_2, and A_3. The variation in the amplitude of the one possible q-function for the Integration domain of Figure 20 is shown in Figure 21. As with 2-D, the *q*-function can be interpreted as virtual displacement of a material point due to the virtual extension of the crack front, $q_t(s)$.

A number of candidate *q* functions for quadratic order elements were presented by Nikishkov and Atluri. Banks-Sills and Sherman studied three of these in some detail (1989). They showed that the *q* functions of Figure 22a and 22b, which arise naturally from the quadratic order shape functions, are inferior to the variation shown in Figure 22c. A similar linear variation of q_t along the crack front can be used for linear order finite elements.

Equation 49 gives the total energy release over the domain of integration for the virtual crack front extension *q*. An approximate local value, $J(s_b)$, can be obtained by normalizing the integral with respect to the area of the virtual crack extension (Koppenhoefer, 1994). That is

$$J(s=b) \approx \frac{\int_{s_a}^{s_b} J(s)\, q_t(s)\, ds}{\int_{s_a}^{s_b} q_t(s)\, ds} = \frac{\bar{J}}{A_q}\,. \tag{50}$$

The modal decomposition approach presented above for 2-D can be extended for 3-D. In this case the decomposed displacement fields are

$$\mathbf{u} = \mathbf{u}^{I} + \mathbf{u}^{II} + \mathbf{u}^{III} = \tfrac{1}{2}\begin{Bmatrix} u_1 + \bar{u}_1 \\ u_2 - \bar{u}_2 \\ u_3 + \bar{u}_3 \end{Bmatrix} + \tfrac{1}{2}\begin{Bmatrix} u_1 - \bar{u}_1 \\ u_2 + \bar{u}_2 \\ 0 \end{Bmatrix} + \tfrac{1}{2}\begin{Bmatrix} 0 \\ 0 \\ u_3 + \bar{u}_3 \end{Bmatrix} \tag{51}$$

with

$$\bar{\mathbf{u}}(x_1, x_2, x_3) = \mathbf{u}(x_1, -x_2, x_3). \tag{52}$$

and the corresponding decomposition of the stress field is

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{I} + \boldsymbol{\sigma}^{II} + \boldsymbol{\sigma}^{III} = \tfrac{1}{2}\begin{bmatrix} \sigma_{11} + \bar{\sigma}_{11} \\ \sigma_{22} + \bar{\sigma}_{22} \\ \sigma_{33} + \bar{\sigma}_{33} \\ \sigma_{12} - \bar{\sigma}_{12} \\ \sigma_{23} - \bar{\sigma}_{23} \\ \sigma_{31} - \bar{\sigma}_{31} \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} \sigma_{11} - \bar{\sigma}_{11} \\ \sigma_{22} - \bar{\sigma}_{22} \\ 0 \\ \sigma_{12} + \bar{\sigma}_{12} \\ 0 \\ 0 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 0 \\ 0 \\ \sigma_{33} - \bar{\sigma}_{33} \\ 0 \\ \sigma_{23} + \bar{\sigma}_{23} \\ \sigma_{31} + \bar{\sigma}_{31} \end{bmatrix}. \tag{53}$$

As with 2-D, the modal decomposition can be performed easily if the crack front mesh is symmetric about the crack plane. Unfortunately, this situation may be difficult to achieve. For the general case of an arbitrarily shaped crack in an arbitrarily shaped object, one usually must use a mesh generator that produces an unstructured tetrahedral mesh with no guarantees of symmetry about the crack plane. Cervenka and Saouma (1997) present an approach where equation 49 is evaluated directly over a cylindrical domain using cylindrical Gauss integration rules. While the simplicity of such an approach is appealing, it suffers from the fact that stresses and displacement derivatives will likely need to be evaluated at locations that are not the optimal sampling points for the elements. This can be overcome somewhat by evaluating the *J*-integral using a virtual regular mesh of hexahedral or wedge elements along the crack front. The displacements for the virtual nodes of these elements are found by interpolating nodal displacements from the real, unstructured, tetrahedral mesh. The stresses and displacement derivatives are then found using the derivatives of the shape functions for the virtual elements. These can be sampled at optimal points for these elements.

3.6 Numerical Examples

In this section a number of simple numerical studies is presented to illustrate, in 2-D, the relative accuracy and convergence properties of displacement correlation, MCCI, and *J*-integral techniques. All three cases are for problems with well-known analytical solutions. The first two studies use the same mesh with differing boundary conditions to model an edge crack plate, Figure 23a, and a center-cracked plate, Figure 23b. The normalized dimensions used in the study are $W/a = 3$ and $h/a = 10$. Plane stress conditions are assumed. A detail of the mesh in the crack region is shown in Figure 24. Quadratic order finite elements were used throughout, and eight quarter-point elements were used at the crack tip in a symmetric pattern.

Convergence behavior studied by reducing the size of the crack tip elements as shown in Figure 25. The results are presented as a function of the ratio of the crack-tip element size to the crack length, *L/a*. This value is used as a convenience, and the present results should not be interpreted as describing any universal relationship between this ratio and the expected accuracy of these methods. In general, the relationship between crack-tip element size and accuracy is complicated and depends on more than just the crack-tip mesh (Saouma, 1984). As with all finite element analysis, one can usually only be assured of accurate results if a convergence study is performed.

For the case of an edge cracked plate, Figure 23a, the analytical expression for the mode I stress intensity factor is

$$K_I = F\sigma\sqrt{a}\text{, with } F = 1.99 - 0.41\left(\frac{a}{W}\right) + 18.7\left(\frac{a}{W}\right)^2 - 38.48\left(\frac{a}{W}\right)^3 + 53.85\left(\frac{a}{W}\right)^4 \qquad (54)$$

The numerical results for four crack-tip element sizes using three different techniques for computing *K*'s are shown in Table 1. The table presents stress intensity factors normalized by the analytical values,

$$I = K_{FEM} / K_{analitical} \text{ .} \qquad (55)$$

One can see from the table that for this case all three methods give accurate SIF values (< 1% error) even for a relatively coarse mesh with relatively large crack-tip elements. The *J*-integral values are the most accurate followed by MCCI and then displacement correlation. The convergence behavior of the three methods is interesting. The *J*-integral approach shows monotonic convergence, while displacement correlation is somewhat more erratic, and MCCI shows nearly uniform values for all mesh sizes.

The stress intensity factor for a center-cracked plate, Figure 23b, is

$$K_I = \sigma\sqrt{\pi a}\sqrt{\sec\frac{\pi a}{W}}\ . \tag{56}$$

The numerical results for this case are presented in Table 2. One can see that that for all three techniques the results are less accurate than those seen in Table 1 (considerably less accurate for displacement correlation). Again, the J-integral results are the most accurate, followed by MCCI and displacement correlation. This time all three methods show monotonic convergence.

In the first two cases only mode I loading is present. The third case involves both mode I and mode II loading. The problem is a center-cracked plate with the crack oriented at an angle to the horizontal axis. This is shown in Figure 26.

The analytical solution for an angled crack in an infinite plate is

$$\begin{aligned} K_I &= \sigma \sin^2 \beta \sqrt{\pi a} \\ K_{II} &= \sigma \sin \beta \cos \beta \sqrt{\pi a} \end{aligned}\ . \tag{57}$$

To approximate the infinite condition in a finite width model, W/a = 80 was used. β was set to 45°. The numerical results are presented in Tables 3 and 4.

In Table 3 one again sees a similar pattern of the relative accuracy among the methods, with all three showing monotonic convergence. The mode II results, Table 4, show a divergence in accuracy for all three methods as the mesh is refined. This implies that a mesh optimized for accurate K_I's may not be the optimal mesh to use for accurate K_{II}'s.

3.7 Considerations for Orthotropic Materials

Many materials of engineering interest are orthotropic in nature. That is, they are anisotropic with three planes of material symmetry. The stress and displacement fields in the crack-tip region, along with the stress-intensity factors and energy release rates, are of interest for fracture mechanics analyses of such materials. Sih, Paris, and Irwin (1965) developed expressions for these quantities for 2-D and some limited 3-D configurations. These are presented in this section. A generalization to additional 3-D configurations is presented by Heinig (1982), but his results are not discussed here.

Sih *et. al.* restrict their analysis to cases where loads in the plane of analysis do not cause out-off-plane displacements. This implies that the plane of analysis (for plane stress and plane strain) and the plane perpendicular to the crack front (for 3-D) are planes of material symmetry. In this case, there is no coupling between the in-plane mode I-II behavior and the anti-plane mode III behavior. In what follows it is assumed (without loss of generality) that the *x-y* plane is the plane of analysis with the crack lying along the *x*-axis. The generalized Hook's law,

$$\{\boldsymbol{\varepsilon}\} = [\mathbf{E}]\{\boldsymbol{\sigma}\} \tag{58}$$

relates the stresses and strains within a linear, anisotropic material. For an orthotropic material with principal material axes (x_1, x_2, and x_3) aligned with the Cartesian (x, y, and z) axes, this can be expressed in terms of engineering constants as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{31} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}. \tag{59}$$

If the in-plane material axes are rotated with respect to the Cartesian coordinates, then the generalized Hook's law can be expressed as

$$\{\boldsymbol{\varepsilon}\} = [\mathbf{T}]^T [\mathbf{E}][\mathbf{T}]\{\boldsymbol{\sigma}\} = [\mathbf{S}]\{\boldsymbol{\sigma}\} \tag{60}$$

or

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & 0 & 0 \\ & S_{22} & S_{23} & S_{24} & 0 & 0 \\ & & S_{33} & S_{34} & 0 & 0 \\ & & & S_{44} & 0 & 0 \\ & & & & S_{55} & S_{56} \\ sym & & & & & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \tag{61}$$

with

$$[\mathbf{T}] = \begin{bmatrix} \cos^2\beta & \sin^2\beta & 0 & \cos\beta\sin\beta & 0 & 0 \\ \sin^2\beta & \cos^2\beta & 0 & -\cos\beta\sin\beta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -2\cos\beta\sin\beta & 2\cos\beta\sin\beta & 0 & \cos^2\beta - \sin^2\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & 0 & 0 & \sin\beta & \cos\beta \end{bmatrix} \tag{62}$$

where β is the angle between the x Cartesian axis and the x_1 material axis. The definitions of the crack tip and material axes are shown in Figure 27.

Equation 61 can be specialized for plane stress ($\sigma_z = \tau_{yz} = \tau_{zx} = 0$) as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{14} \\ & S_{22} & S_{24} \\ sym & & S_{44} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}. \tag{63}$$

For plane strain ($\varepsilon_z = \tau_{yz} = \tau_{zx} = 0$) the expression is

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} - \frac{S_{12}S_{13}}{S_{33}} & S_{12}\left(1 - \frac{S_{23}}{S_{33}}\right) & S_{14} - \frac{S_{12}S_{34}}{S_{33}} \\ & S_{22}\left(1 - \frac{S_{23}}{S_{33}}\right) & S_{24} - \frac{S_{22}S_{34}}{S_{33}} \\ sym & & S_{44} - \frac{S_{22}S_{34}}{S_{33}} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} S'_{11} & S'_{12} & S'_{14} \\ & S'_{22} & S'_{24} \\ sym & & S'_{44} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{64}$$

The plane stress and plane strain expressions are structurally similar, the difference being the values of the constitutive coefficients. Therefore, it should be understood that the plane stress development that follows is valid for plane strain analyses if S'_{ij} coefficients are substituted for S_{ij} in all expressions.

From Sih *et. al.* the in-plane stress and displacement fields near a crack tip can be expressed in polar coordinates, r-θ (see Figure 27), as

$$\begin{aligned}
\sigma_x &= \frac{K_I}{\sqrt{2\pi r}} \mathrm{Re}\left[\frac{\mu_1 \mu_2}{\mu_1 - \mu_2}\left(\frac{\mu_2}{d_2} - \frac{\mu_1}{d_1}\right) + \frac{K_{II}}{K_I}\frac{1}{\mu_1 - \mu_2}\left(\frac{\mu_2^2}{d_2} - \frac{\mu_1^2}{d_1}\right)\right] \\
\sigma_y &= \frac{K_I}{\sqrt{2\pi r}} \mathrm{Re}\left[\frac{1}{\mu_1 - \mu_2}\left(\frac{\mu_1}{d_2} - \frac{\mu_2}{d_1}\right) + \frac{K_{II}}{K_I}\frac{1}{\mu_1 - \mu_2}\left(\frac{1}{d_2} - \frac{1}{d_1}\right)\right] \\
\tau_{xy} &= \frac{K_I}{\sqrt{2\pi r}} \mathrm{Re}\left[\frac{\mu_1 \mu_2}{\mu_1 - \mu_2}\left(\frac{1}{d_1} - \frac{1}{d_2}\right) + \frac{K_{II}}{K_I}\frac{1}{\mu_1 - \mu_2}\left(\frac{\mu_1}{d_1} - \frac{\mu_2}{d_1 2}\right)\right]
\end{aligned} \tag{65}$$

and

$$\begin{aligned}
u &= K_I \sqrt{\frac{2r}{\pi}}\, \mathrm{Re}\left[\frac{1}{\mu_1 - \mu_2}\left(\mu_1 p_2 d_2 - \mu_2 p_1 d_1 + \frac{K_{II}}{K_I}\left[p_2 d_2 - p_1 d_1\right]\right)\right] \\
v &= K_I \sqrt{\frac{2r}{\pi}}\, \mathrm{Re}\left[\frac{1}{\mu_1 - \mu_2}\left(\mu_1 q_2 d_2 - \mu_2 q_1 d_1 + \frac{K_{II}}{K_I}\left[q_2 d_2 - q_1 d_1\right]\right)\right]
\end{aligned} \tag{66}$$

with

$$\begin{aligned}
d_i &= \sqrt{\cos\theta + \mu_i \sin\theta} \\
p_i &= S_{11}\mu_i^2 + S_{12} - S_{14}\mu_i \quad . \\
q_i &= S_{12}\mu_i + S_{22}/\mu_i - S_{24}
\end{aligned} \tag{67}$$

μ_1 and μ_2 are the two roots with positive imaginary parts of the equation

$$S_{11}\mu^4 - 2S_{14}\mu^3 + 2(S_{12} + S_{44})\mu^2 - 2S_{24}\mu + S_{22} = 0 \tag{68}$$

(Since this equation has real coefficients, the roots will be two sets of complex conjugates).

Energy release rates for this case can be derived by substituting the stress and displacement expressions of 65 and 66 into Irwin's crack closure integral (Irwin 1957)

$$G = \lim_{\Delta L \to 0} \frac{1}{\Delta L} \int_0^{\Delta L} \sigma_{2i}(\Delta L - r, \theta = 0)\, u_i(r, \theta = \pi)\, dr, \quad (i = 1,2)\,. \tag{69}$$

This yields, separated by modes, the following expressions:

$$\begin{aligned} G_I &= -\tfrac{1}{2} K_I S_{22}\, \mathrm{Im}\left[\frac{K_I(\mu_1 + \mu_2) + K_{II}}{\mu_1\mu_2}\right] \\ G_{II} &= \tfrac{1}{2} K_{II} S_{11}\, \mathrm{Im}\left[K_{II}(\mu_1 + \mu_2) + K_I \mu_1\mu_2\right] \end{aligned} \quad . \tag{70}$$

In a finite element context, the energy release rates can be computed with the *J*-integral approach (Section 3, above). Equations 70 can be solved for the stress intensity factors

$$\begin{aligned} K_I &= \pm \frac{A_2\left[2G_{II}S_{22}(A_0A_2 - A_1^2) - (G_I S_{11}A_3A_1^2 + G_{II}S_{22}(2A_0A_2 - A_1^2) \pm A_4)\right]}{A_3\left[S_{11}S_{22}A_2(A_0A_2 - A_1^2)(G_I S_{11}A_3A_1^2 + G_{II}S_{22}(2A_0A_2 - A_1^2) \pm A_4)\right]^{1/2}} \\ K_{II} &= \pm \frac{\left[S_{11}S_{22}A_2(A_0A_2 - A_1^2)(G_I S_{11}A_3A_1^2 + G_{II}S_{22}(2A_0A_2 - A_1^2) \pm A_4)\right]^{1/2}}{S_{11}S_{22}A_2(A_0A_2 - A_1^2)} \end{aligned} \tag{71}$$

with

$$\begin{aligned}
A_0 &= \text{Im}(\mu_1)[\text{Im}(\mu_2)\,\text{Re}(\mu_2)]^2 + \text{Im}(\mu_2)[\text{Im}(\mu_1)\,\text{Re}(\mu_1)]^2 \\
A_1 &= \text{Re}(\mu_1)\,\text{Im}(\mu_2) + \text{Im}(\mu_1)\,\text{Re}(\mu_2) \\
A_2 &= \text{Im}(\mu_1) + \text{Im}(\mu_2) \\
A_3 &= [\text{Re}(\mu_1)\,\text{Re}(\mu_2)]^2 + [\text{Im}(\mu_1)\,\text{Im}(\mu_2)]^2 + [\text{Re}(\mu_1)\,\text{Im}(\mu_2)]^2 + [\text{Im}(\mu_1)\,\text{Re}(\mu_2)]^2 \\
A_4 &= A_1\left[(G_I S_{11} A_1 A_3)^2 + 2G_I G_{II} S_{11} S_{22}(2A_0 A_2 - A_1^2 A_3) + (G_{II} S_{22} A_1)^2\right]^{1/2}
\end{aligned} \tag{72}$$

The signs used for the full expressions and the A_4 terms of equation 71 must be determined from the signs of the crack opening displacement (COD) and crack sliding displacement (CSD). These signs are given in Table 5. For example, if the crack is opening (COD is positive) and the sliding displacement is negative, then negative signs should be used for both the full expressions in equation 71 and the A_4 terms.

For the pure antiplane shear (mode III) case

$$u = v = 0, \quad w = w(x, y) \tag{73}$$

Hook's law for this case is

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} S_{55} & S_{56} \\ sym & S_{66} \end{bmatrix} \begin{Bmatrix} \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{1}{S_{55}S_{66} - S_{56}^2} \begin{bmatrix} S_{66} & -S_{56} \\ sym & S_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}. \tag{74}$$

The stress and displacement fields are

$$\tau_{yz} = \frac{K_{III}}{\sqrt{2\pi r}} \operatorname{Re}\left[\frac{1}{\sqrt{\cos\theta + \mu_3 \sin\theta}}\right]$$
$$\tau_{zx} = \frac{K_{III}}{\sqrt{2\pi r}} \operatorname{Re}\left[\frac{\mu_3}{\sqrt{\cos\theta + \mu_3 \sin\theta}}\right] \tag{75}$$

and

$$w = K_{III}\sqrt{\frac{2r}{\pi}} \operatorname{Re}\left[\frac{\left(S_{55}S_{66} - S_{56}^2\right)\sqrt{\cos\theta + \mu_3 \sin\theta}}{\mu_3 S_{66} - S_{56}}\right] \tag{76}$$

where μ_3 is the root of the equation

$$\frac{S_{66}\mu_3^2 - 2S_{56}\mu_3 + S_{55}}{S_{55}S_{66} - S_{56}^2} = 0 \tag{77}$$

with the positive imaginary part.

Again, using Irwin's crack closure integral, equation 69, the expression for the energy release rate is

$$G_{III} = \tfrac{1}{2} K_{III}^2 \frac{\operatorname{Im}\left[\mu_3 S_{66} - S_{56}\right]}{S_{55}S_{66}}, \tag{78}$$

that can be solved simply for the stress intensity factor

$$K_{III} = \sqrt{\frac{2G_{III} S_{55} S_{66}}{\operatorname{Im}\left[\mu_3 S_{66} - S_{56}\right]}}\,. \tag{79}$$

In summary, Sih, Paris, and Irwin developed expressions for the near tip-stress and displacement fields for cracks in orthotropic materials where one of the planes of material symmetry is perpendicular to the crack front, the plane of analysis (equations 65 and 66). Irwin's crack closure integral, 69, can be used to relate the stress intensity factors to the energy release rate, which for an LEFM analysis can be computed using finite elements and the *J*-integral technique (Sections 3.4-3.5, above).

3.8 Considerations for Plate and Shell Finite Elements

There are many practical applications of fracture mechanics to structures that can be idealized as thin plates or shells. For example, a typical narrow body transport aircraft has a fuselage radius of about 2 m (79 inches). However, the skin thickness is only about 1 mm (0.4 inches). Clearly, with a radius to thickness ratio of 2000, thin shell assumptions are valid and economies can be realized in finite element analysis if thin shell elements are used rather than modeling the fuselage with 3D elements (Potyondy, 1995).

In this section, the stress intensity factors for through cracks in plates that correspond to Kirchoff and Reissner plate theories are presented. It is argued that the Kirchoff assumptions are more appropriate for most finite element analyses. Expressions for extracting the stress intensity factors using a crack closure approach are then presented. Much of the material presented in this section follows very closely the work of Hui and Zehnder (1992).

A naive fracture analysis approach for plate and shell structures is to assume 2-D (plane stress) behavior because the plate is flat or the radius of curvature of the shell is very large. This can lead to highly inaccurate (and usually unconservative) predictions. A 2-D analysis ignores out-of-plane bending which usually is the most flexible mode of deformation. For example, with aircraft fuselages or other pressure vessels, the internal pressure causes the crack faces to "bulge". This bulging behavior, illustrated in Figure 28, significantly increases the stress intensity factors. Furthermore, the bulging effect is nonlinear. For small amounts of bulging the structure is very flexible for out-of-plane bending of the crack flanks. However, as the amount of bulging increases the fibers parallel to the crack faces "stress stiffen", resisting further bulging.

When structures are idealized as thin plates or shells it is normal to assume that the overall stresses and displacements are a combination of membrane behavior that is constant through the thickness, and bending behavior that has a through thickness variation. Similarly, membrane and bending fracture modes can be identified. These are shown in Figure 29. The symmetric and anti-symmetric membrane modes correspond to the plane stress modes I and II. The bending behavior can be separated into two modes also, corresponding to symmetric and anti-symmetric bending about the crack.

All plate theories make assumptions about the nature of the stress and displacement variation through the plate thickness. Two different plate theories, Kirchoff theory (Timoshenko 1959) and Reissner theory (1947) are considered here. Each results in different bending stress intensity factors. Stress intensity factors based on Kirchoff bending are denoted k_1 and k_2. Stress intensity factors based on Reissner theory are denoted K_1, K_2, and K_3.

The stress field near the tip of a through crack in a thin elastic plate was first obtained by Williams (1961) using Kirchhoff assumptions. With respect to the coordinate system shown in Figure 30, these are:

$$\begin{pmatrix} \sigma_{rr} \\ \sigma_{r\theta} \\ \sigma_{\theta\theta} \end{pmatrix} = \frac{k_1}{(3+\nu)\sqrt{2r}} \frac{z}{2h} \begin{pmatrix} (3+5\nu)\cos(\theta/2) - (7+\nu)\cos(3\theta/2) \\ -(1-\nu)\sin(\theta/2) + (7+\nu)\sin(3\theta/2) \\ (5+3\nu)\cos(\theta/2) + (7+\nu)\cos(3\theta/2) \end{pmatrix} + \frac{k_2}{(3+\nu)\sqrt{2r}} \frac{z}{2h} \begin{pmatrix} (3-5\nu)\sin(\theta/2) + (5+3\nu)\sin(3\theta/2) \\ -(1-\nu)\cos(\theta/2) + (5+3\nu)\cos(3\theta/2) \\ -2(5+3\nu)\cos(\theta/2)\sin(\theta) \end{pmatrix}, \quad (80)$$

$$\begin{pmatrix} \sigma_{rz} \\ \sigma_{\theta z} \end{pmatrix} = \frac{\left[1 - \left(\frac{2z}{h}\right)^2\right] \frac{h}{2}}{(3+\nu)(2r)^{3/2}} \begin{pmatrix} -k_1\cos(\theta/2) + k_2\sin(\theta/2) \\ -k_1\sin(\theta/2) - k_2\cos(\theta/2) \end{pmatrix}, \quad (81)$$

and

$$\sigma_{zz} = 0,$$

were ν is the Poisson's ratio, h is the plate thickness, and k_1, k_2 are the stress intensity factors. The in-plane stress components, σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$, have an $r^{-1/2}$ singularity, which is the same as predicted by elasticity theory. The out-of-plane shear stresses, σ_{rz} and $\sigma_{\theta z}$, have an $r^{-3/2}$ singularity. Unlike elasticity theory, the angular variation of the stress fields depends on the Poisson's ratio. The discrepancy between the stress singularity between elasticity theory and the plate theory arises from the inability of the Kirchhoff theory to completely satisfy stress free boundary conditions along the crack face.

The inconsistency in the order of the singularity for the out-of-plane shear stresses has led a number of authors to investigate the near-tip stress fields using Reissner plate theory (Knowles, 1960; Hartranft, 1968; Wang, 1970; Joseph, 1991). The resulting asymptotic crack-tip stress field is

$$\begin{pmatrix} \sigma_{rr} \\ \sigma_{r\theta} \\ \sigma_{\theta\theta} \end{pmatrix} = \frac{K_1}{\sqrt{2r}} \frac{z}{2h} \begin{pmatrix} 5\cos(\theta/2) - \cos(3\theta/2) \\ \sin(\theta/2) + \sin(3\theta/2) \\ 3\cos(\theta/2) + \cos(3\theta/2) \end{pmatrix} + \frac{K_2}{\sqrt{2r}} \frac{z}{2h} \begin{pmatrix} -5\sin(\theta/2) + 3\sin(3\theta/2) \\ \cos(\theta/2) + 3\cos(3\theta/2) \\ -3\sin(\theta/2) - 3\sin(3\theta/2) \end{pmatrix}, \tag{82}$$

$$\begin{pmatrix} \sigma_{rz} \\ \sigma_{\theta z} \end{pmatrix} = \frac{K_3}{\sqrt{2r}} \left[1 - \left(\frac{2z}{h} \right)^2 \right] \begin{pmatrix} \sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}, \tag{83}$$

With the exception of the *z* variation, these crack-tip stress fields are identical to those predicted by elasticity theory.

With two separate sets of relations for the stress intensity factors and near tip stresses for cracks in plates, two questions arise: is there a relationship between them, and which set is more appropriate for use in a finite element analysis? Hui and Zehnder (1992) used evaluations of the J-integral for the two crack-tip fields to demonstrate that there is a universal relationship between the two sets of stress intensity factors that is independent of the specimen geometry and applied loading. These relationships are

$$K_1 / k_1 = \left[(1+\nu)/(3+\nu) \right]^{1/2} \tag{84}$$

and

$$k_2^2 \frac{1+\nu}{3+\nu} = K_2^2 + K_3^2 \frac{8(1+\nu)}{5} \,. \tag{85}$$

The implication of this relationship is significant. From a finite element analysis point-of –view, the important question is which plate theory should be used. The Reissner theory yields the expected singularity at the crack tip for all stress components. However, Hui and Zender argue that the region of dominance of the Reissner K field is ~ h/10. This is smaller than the expected plastic zone size for many engineering materials. The region of dominance of the Kirchhoff field is ~ L/10, where, similar to plane stress, L is a relevant in-plane dimension, such a crack length. Equations 84 and 85 demonstrate that the stress intensity factors in the Reissner *K*-dominant zone are uniquely determined by the stress intensity factors in the much larger Kirchoff zone. This is similar to the concept of small-scale yielding in LEFM, where the size of the plastic zone is small compared to the region of dominance of the elastic *K* field. The behavior in the small-scale yielding zone is determined by the enclosing *K* field. This is illustrated schematically in Figure 31.

Therefore, for finite element analysis in most cases the Kirchhoff theory should be preferred, as the size of the region of dominance of this theory is similar to what one would expect for plane stress. This means that one could expect to be able to extract accurate Kirchoff stress intensity factors using "reasonably" sized elements relative to the crack length (unlike the Reissner theory, where element sizes on the order of the plate thickness would be required). Furthermore, if needed, the Reissner stress intensity factors can be determined from the Kirchhoff stress intensity factors.

Moreover, both theories assume that the crack front is oriented perpendicular to the thickness of the crack. For many materials, metals in particular, this assumption is frequently violated, with "slant cracking" being typical. Obviously, deviation from the crack orientations assumption will have a much larger impact on the Reissner predictions, with a characteristic dimension of the plate thickness, than on the Kirchhoff predications where the characteristic dimension is much larger, on the order of the crack length.

Among the important interim results in Hui and Zehnder's analysis are expressions for the relationship between the energy release rates and the Kirchhoff stress intensity factors:

$$G_1 = \frac{k_1^2 \pi (1+\nu)}{3E(3+\nu)} \quad \text{and} \quad G_2 = \frac{k_2^2 \pi (1+\nu)}{3E(3+\nu)} \tag{86}$$

These can be used in combination with the plane stress expressions,

$$G_I = \frac{K_I^2}{E} \quad \text{and} \quad G_{II} = \frac{K_{II}^2}{E}. \tag{87}$$

to extract stress intensity factors from thin shell analyses, provided that the energy release rates can be computed from a finite element analysis.

The Modified Crack Closure Integral was presented in Section 3.3. There it was shown that the energy released during crack growth is equal to the work done by the tractions acting over the area of crack extension. Switching, for the moment, to the notation that x_1, x_2, x_3, = x, y, z, for a linear elastic plate of thickness h, the energy release rate for a self-similar extension of a through-crack lying in the x-z plane is (Irwin 1957)

$$G = \lim_{\Delta L \to 0} \frac{1}{2h\Delta L} \int_0^{\Delta L} \int_{-h/2}^{+h/2} \sigma_{2i}(x_1, \theta = 0) \Delta u_i (\Delta L - x_1, \theta = \pi)\, dx_3 dx_1 , \qquad (88)$$

where Δu_i are the components of relative crack tip displacements for a crack extension of ΔL, and the repeated index i implies summation over i = 1, 2, 3.

Using the above crack closure integral, but segregating the displacements and stresses associated with each mode of fracture, Viz *et. al.* (1995) developed expressions for the energy release rates in terms of nodal (generalized) forces and displacements. This was done for plate and shell finite elements with linear shape functions. Figure 32a illustrates the procedure using two finite element analyses. The forces and moments are extracted at node 1 (the crack tip) from the first analysis. The crack is then extended and the model is reanalyzed. The displacements and rotations are extracted from nodes 2 and 3 of the second analysis. The expressions for the (Kirchoff) energy release rates are

$$\begin{aligned} G_I &= \tfrac{1}{2}\left[F_y^1(u_y^2 - u_y^3) + M_z^1(\theta_z^2 - \theta_z^3)\right] \\ G_{II} &= \tfrac{1}{2}\left[F_x^1(u_x^2 - u_x^3)\right] \\ G_1 &= \tfrac{1}{2}\left[M_x^1(\theta_x^2 - \theta_x^3)\right] \\ G_2 &= \tfrac{1}{2}\left[F_z^1(u_z^2 - u_z^3) + M_y^1(\theta_y^2 - \theta_y^3)\right] \end{aligned} \qquad (89)$$

where for the forces and displacements a superscript denotes the node number and a subscript denotes the direction (z is the out-of-plane direction). As with the planar analysis case, a similar computation can be performed using the results of only one analysis. The nodes used for this case are illustrated in Figure 32b.

In summary, for through-cracks in thin plates and shells there are two membrane and two bending fracture modes. The stress intensity factors for the membrane modes correspond to those of plane stress modes I and II. For bending, the definition of the stress intensity factors depends on the plate theory used. Usually Kirchoff plate theory is more appropriate for thin shell finite element analysis because the zone of dominance is on the order of the crack length, whereas the zone of dominance is on the order of the plate thickness for Reissner theory. Expressions for extracting energy release rates from a finite element analysis are given in equation 89. These can be used with expressions 86 and 87 to compute stress intensity factors.

3.9 Summary

A number of different techniques for extracting stress intensity factors from finite element results were presented in this section. Of these, the J-integral approach is the most accurate and should be used preferentially. Unfortunately, the implementation of the method is the most involved of those shown, and is aided if one has access to the finite element programs subroutines for shape functions and numerical integration (often inaccessible to users of a commercial finite element program). Obviously, these capabilities can be replicated in a stand-alone post-processing program for computing SIF's but often acceptably accurate results can be obtained using the MCCI approach that uses only nodal displacements and forces, which are standard outputs from most FEM programs.

The displacement correlation technique is the least accurate but is simple enough that it is readily amenable to hand calculations. Also, because it does not require additional terms for cases with crack-face tractions or body forces, it provides a simple "sanity" check to make sure that the more accurate techniques are formulated and being used properly.

This section also described how some of these extraction techniques are used for cases in which the cracking material is anisotropic, and also for cases in which the cracking structure can best be modeled with plate or shell elements.