

CEE 770 Meeting 7

Objectives of This Meeting

Learn Generalized Stiffness Derivative Technique (aka Generalized Virtual Crack Extension, VCE, techniques) for:

- Computing G , and its derivatives
- Using these values to predict single crack tip stability
- Using these values to predict multiple crack tip stability
- Using these values to predict shape evolution of a 3D crack

Review Pages 72-74, VCE/Stiffness Derivative Methods

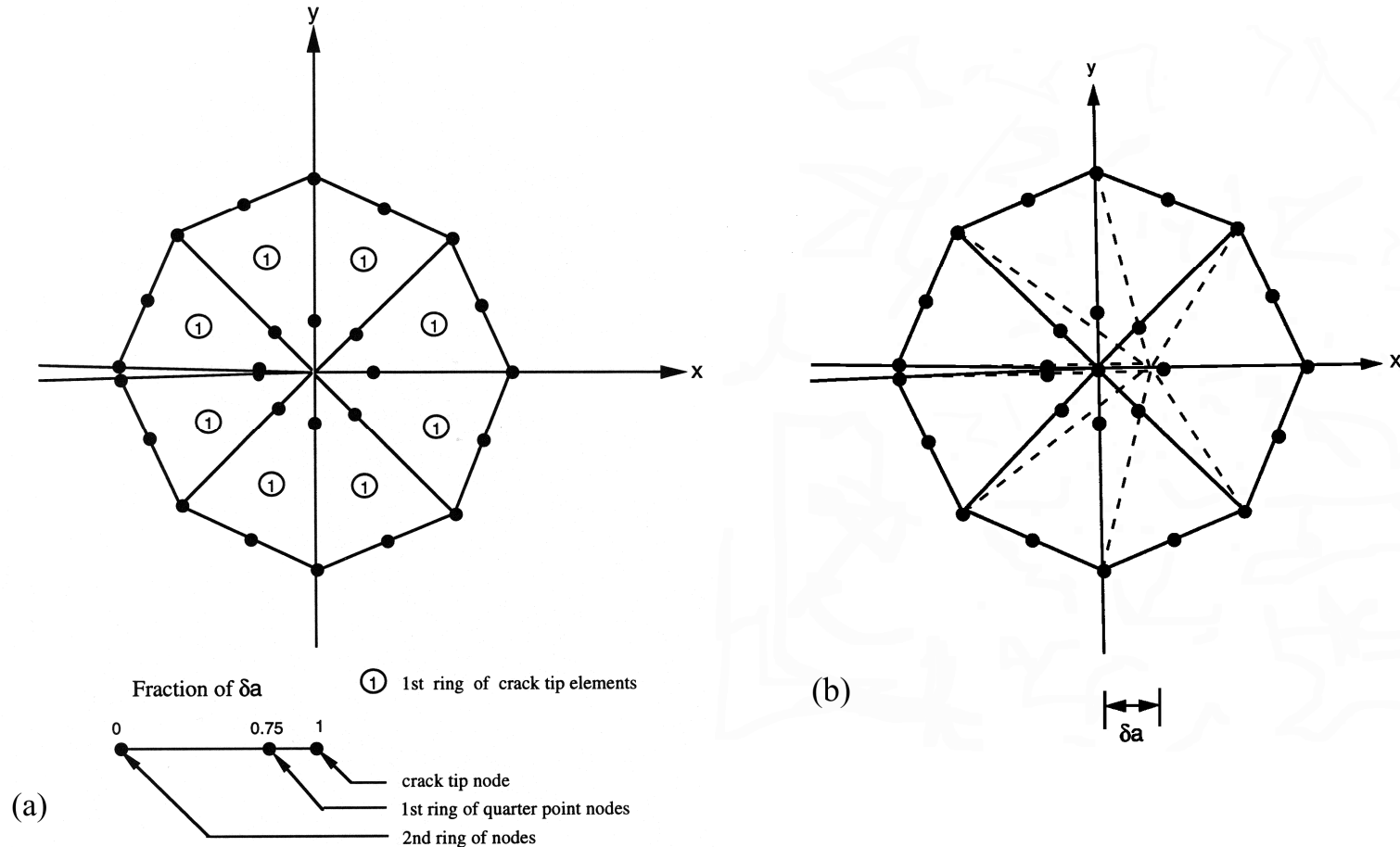
Recall from your studies of LEFM theory that energy release rate (aka crack driving force) for a single 2D crack tip, in FEM context, is defined as:

$$\Pi = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{f} \quad (43, \text{p.72})$$

$$G \equiv -\frac{\partial \Pi}{\partial a} = -\frac{1}{2} \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial a} \mathbf{u} + \mathbf{u}^T \frac{\partial \mathbf{f}}{\partial a} \quad (44, \text{page 73, from Parks, 1975})$$

Where a is the length of crack, \mathbf{u} is the nodal displacement vector, \mathbf{K} is the global stiffness matrix, \mathbf{f} the global force vector, and nonzero contributions to $\delta \mathbf{K} / \delta a$ and $\delta \mathbf{f} / \delta a$ occur only over elements adjacent to the crack front.

Typical Near-Tip Meshing for a Virtual Crack Extension, δa

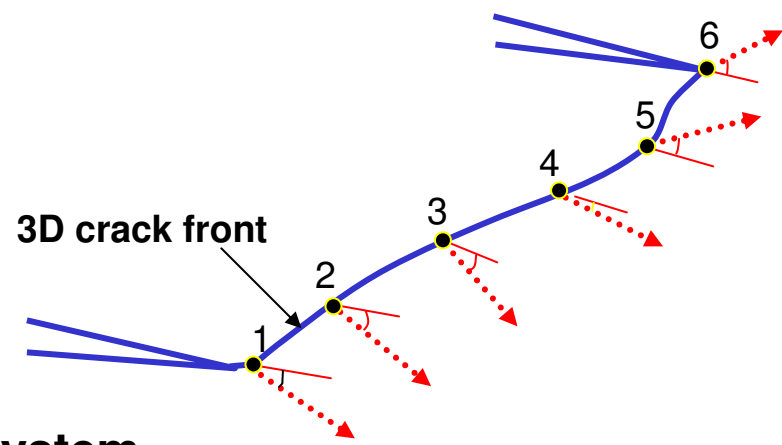
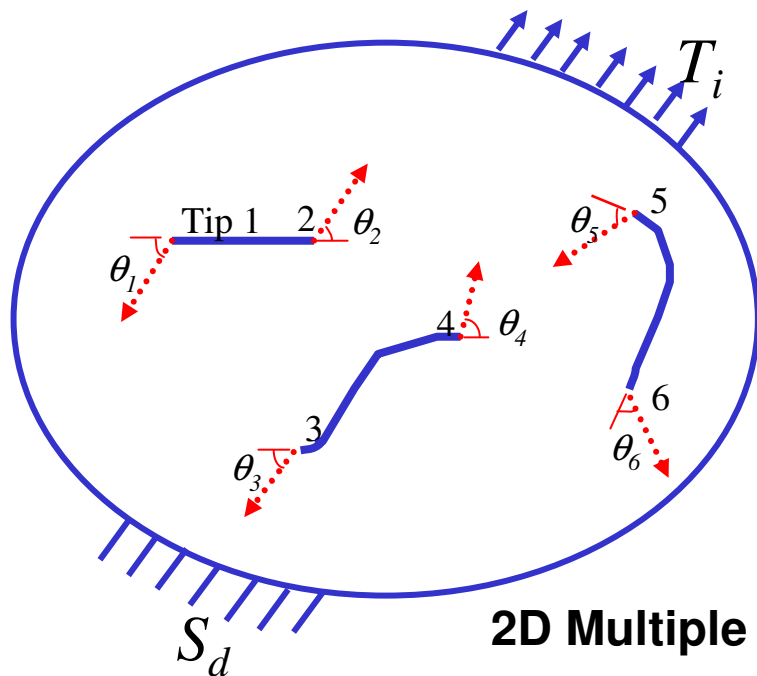


Energy Release Rate and Its Rates

$$G \equiv -\frac{\partial \Pi}{\partial a} = -\frac{1}{2} \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial a} \mathbf{u} + \mathbf{u}^T \frac{\partial \mathbf{f}}{\partial a} \quad (44, \text{ page 73, from Parks, 1975})$$

This is OK for a single, 2D crack tip, **BUT** for more general, multiple crack 2D case, and always for 3D:

$$G_i \equiv -\frac{\delta \Pi}{\delta a_i} = -\frac{1}{2} \mathbf{u}^T \frac{\delta \mathbf{K}}{\delta a_i} \mathbf{u} + \mathbf{u}^T \frac{\delta \mathbf{f}}{\delta a_i} \quad (69)$$



Energy Release Rate and Its Rates: Key Issues

- How to compute the derivatives in

$$G_i \equiv -\frac{\delta\Pi}{\delta a_i} = -\frac{1}{2} \mathbf{u}^T \frac{\delta\mathbf{K}}{\delta a_i} \mathbf{u} + \mathbf{u}^T \frac{\delta\mathbf{f}}{\delta a_i}$$

accurately?

- What is the meaning of δa for 3D crack?
- How to compute higher order derivatives? And why compute them?

When **Can** a 2D Crack Propagate?

When **Can** a 3D Crack Change Shape?

For a 2D system with a **single crack**, the necessary and **sufficient** conditions for crack extension are:

$$G = G_c, \text{ and} \quad \delta G / \delta a \geq 0 \quad \text{Why????} \quad (70)$$

where,

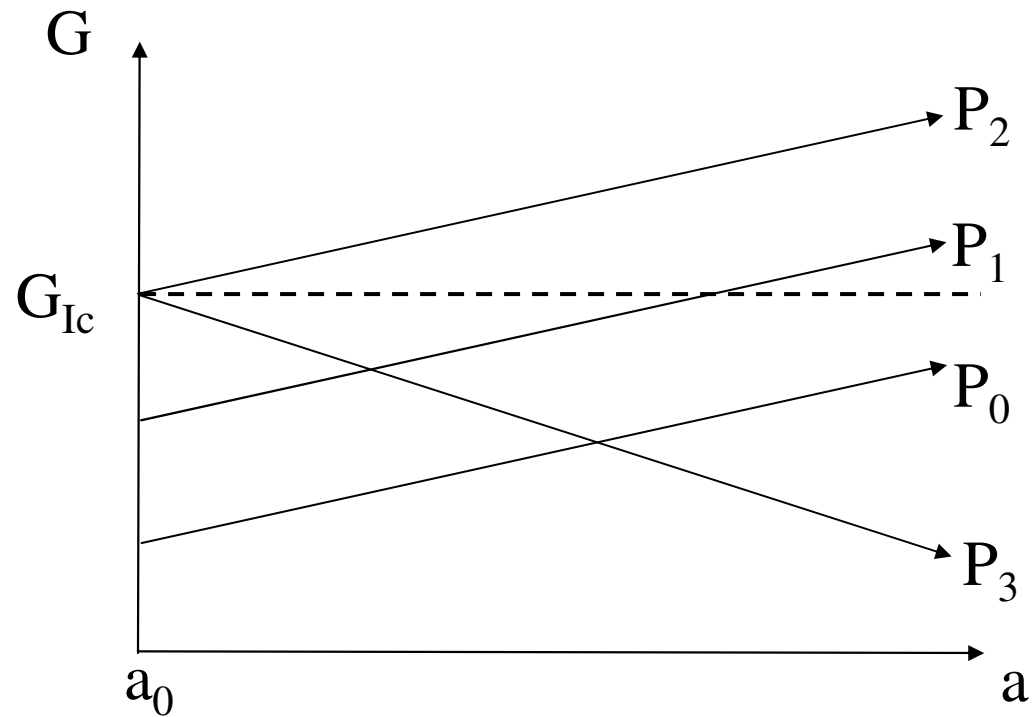
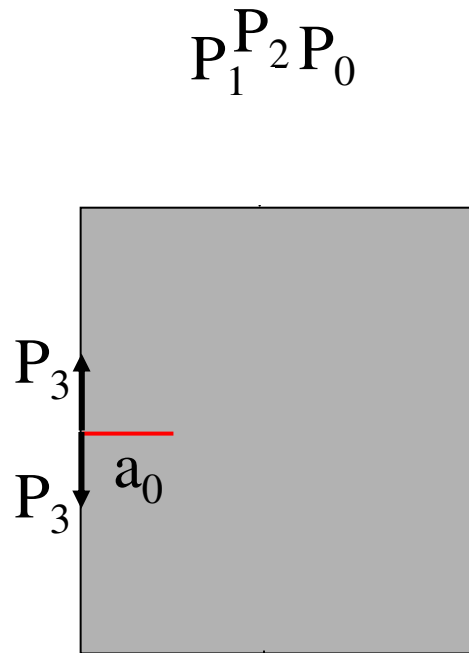
$$G = - \delta \Pi / \delta a \quad \Pi = \text{Potential energy of the system}$$

$$\delta G / \delta a = - \delta^2 \Pi / \delta a^2$$

Therefore, one needs to be able to calculate both G and $\delta G / \delta a$ accurately.

Moreover, in many 2D situations it is possible that the system will have multiple cracks!

The Sufficiency Condition of LEFM



P_1 P_2 P_0

The Generalized Virtual Crack Extension Method

Hwang, Wawrzynek, Tayebi, Ingraffea. On the Virtual Crack Extension Method for Calculation of the Rates of Energy Release Rate.
Eng. Fract. Mech., **59**, 521-542, 1998

Hwang C G, Wawrzynek P A, Ingraffea A R. On the calculation of derivatives of stress intensity factors for multiple cracks, *Eng. Fract. Mech.*, **72**, 1171-1196, 2005.

Provides G_i and $\delta G_i / \delta a_i$ for multiple crack systems....

...subjected to arbitrary **thermal loading, crack-face loading, body forces**, in 2D/3D/axisymmetric problems, from a **single FEA**.

$$G_i \equiv -\frac{\delta \Pi}{\delta a_i} = -\frac{1}{2} \mathbf{u}^T \frac{\delta \mathbf{K}}{\delta a_i} \mathbf{u} + \mathbf{u}^T \frac{\delta f}{\delta a_i}$$

Non-Zero for these loadings

$$\frac{\delta G_i}{\delta a_j} = -\mathbf{u}^T \frac{\delta \mathbf{K}}{\delta a_i} \frac{\delta \mathbf{u}}{\delta a_j} - \frac{1}{2} \mathbf{u}^T \frac{\delta^2 \mathbf{K}}{\delta a_i \delta a_j} \mathbf{u} + \frac{\delta \mathbf{u}}{\delta a_j}^T \frac{\delta f}{\delta a_i} + \mathbf{u}^T \frac{\delta^2 f}{\delta a_i \delta a_j}$$

Null for 2D
Non-Zero for 3D

when $i \neq j$

(71)

How to Compute The Derivatives Analytically and Accurately

For a multiply cracked system, the following expressions are the rates of energy release rate for crack tip i . They are the analytical and generalized version of the stiffness derivative technique (Parks, 1975). Even more general expressions which account for mixed-mode are also available in Hwang *et al.*, 1998:

$$i \neq j$$

$$\frac{\delta G_i}{\delta a_j} = -u^T \frac{\delta K}{\delta a_i} K^{-1} \left(\frac{\delta f}{\delta a_j} - \frac{\delta K}{\delta a_j} u \right) + \left(\frac{\delta f}{\delta a_j} - \frac{\delta K}{\delta a_j} u \right)^T K^{-1T} \frac{\delta f}{\delta a_i} \quad (72)$$

$$i = j$$

$$\frac{\delta G_i}{\delta a_i} = -u^T \frac{\delta K}{\delta a_i} K^{-1} \left(\frac{\delta f}{\delta a_i} - \frac{\delta K}{\delta a_i} u \right) - \frac{1}{2} u^T \frac{\delta^2 K}{\delta a_i^2} u + \left(\frac{\delta f}{\delta a_i} - \frac{\delta K}{\delta a_i} u \right)^T K^{-1T} \frac{\delta f}{\delta a_i} + u^T \frac{\delta^2 f}{\delta a_i^2} \quad (73)$$

How to Compute The Derivatives Analytically and Accurately

Note that:

1. These operations need only be performed on a small number of elements around the crack tip.
2. The element stiffness derivatives can be computed analytically via:

$$\delta k = \int_v \left[\delta B^T D B + B^T D \delta B + Tr(\tilde{\epsilon}) B^T D B \right] dV \quad (74)$$

$$\delta^2 k = \int_v \left[\delta^2 B^T D B + 2 \delta B^T D \delta B + B^T D \delta^2 B + 2 \tilde{\epsilon} B^T D B + 2 Tr(\tilde{\epsilon}) (\delta B^T D B + B^T D \delta B) \right] dV \quad (75)$$

where $\tilde{\epsilon}$ is the virtual strain-like matrix,

$$\tilde{\epsilon} = J^{-1} \begin{Bmatrix} \partial N / \partial \xi^1 \\ \partial N / \partial \xi^2 \end{Bmatrix} \begin{bmatrix} \Delta_n^1 & \\ & \Delta_n^2 \end{bmatrix} = \begin{bmatrix} \tilde{\epsilon}_{11} & \tilde{\epsilon}_{12} \\ \tilde{\epsilon}_{21} & \tilde{\epsilon}_{22} \end{bmatrix} \quad (76)$$

where Δ 's are the geometry changes of the meshes due to virtual crack extension.

How to Compute The Derivatives Analytically and Accurately

$$\frac{\delta^2 G_i}{\delta a_i^2} = -\frac{1}{2} u^T \frac{\delta^3 K}{\delta a_i^3} u - 2 u^T \frac{\delta^2 K}{\delta a_i^2} \frac{\delta u}{\delta a_i} - u^T \frac{\delta K}{\delta a_i} \frac{\delta^2 u}{\delta a_i^2} \quad (77)$$

$$- \frac{\delta u^T}{\delta a_i} \frac{\delta K}{\delta a_i} \frac{\delta u}{\delta a_i} + \frac{\delta^2 u^T}{\delta a_i^2} \frac{\delta f}{\delta a_i} + 2 \frac{\delta u^T}{\delta a_i} \frac{\delta^2 f}{\delta a_i^2} + u^T \frac{\delta^3 f}{\delta a_i^3}$$

$$\frac{\delta u}{\delta a_i} = K^{-1} \left(\frac{\delta f}{\delta a_i} - \frac{\delta K}{\delta a_i} u \right) \quad \text{and} \quad \frac{\delta^2 u}{\delta a_i^2} = K^{-1} \left(\frac{\delta^2 f}{\delta a_i^2} - \frac{\delta^2 K}{\delta a_i^2} u - 2 \frac{\delta K}{\delta a_i} \frac{\delta u}{\delta a_i} \right)$$

$$\delta^3 k = \int_v \left[\delta^3 B^T D B + 3 \delta^2 B^T D \delta B + 3 \delta B^T D \delta^2 B + B^T D \delta^3 B \right] dV$$

$$+ \int_v \left[2 |\tilde{\epsilon}^2| B^T D B + 6 |\tilde{\epsilon}| (\delta B^T D B + B^T D \delta B) + 2 |\tilde{\epsilon}| \text{Tr}(\tilde{\epsilon}) B^T D B \right] dV$$

$$\delta^3 B = -6 (\tilde{\epsilon})^3 B + 3 \int_v \left[\delta^2 B^T D B + 2 \delta B^T D \delta B + B^T D \delta^2 B \right] \text{Tr}(\tilde{\epsilon}) dV \quad (78)$$

How to Compute The Derivatives Analytically and Accurately

You also need the force derivatives when relevant, assembled from elemental derivatives from the virtually perturbed elements:

$$\begin{aligned}\frac{\delta f}{\delta a} &= \sum_e \frac{\delta f_e}{\delta a} \\ \frac{\delta^2 f}{\delta a^2} &= \sum_e \frac{\delta^2 f_e}{\delta a^2}\end{aligned}\tag{79}$$

For example, if there is crack face pressure, p

$$\delta f_e = \delta \int_s N^T p \, ds = \int_s \left[N^T \delta p + Tr(\tilde{\epsilon}) N^T p \right] ds\tag{80}$$

$$\delta^2 f_e = \delta^2 \int_s N^T p \, ds = \int_s \left[N^T \delta^2 p + 2Tr(\tilde{\epsilon}) N^T \delta p + 2|\tilde{\epsilon}| N^T p \right] ds$$

If You Want Stress Intensity Rather Than Energy Release Rate Derivatives...

Then, where $H = E$ for plane stress and $H = E/(1 - \nu^2)$ for plane strain:

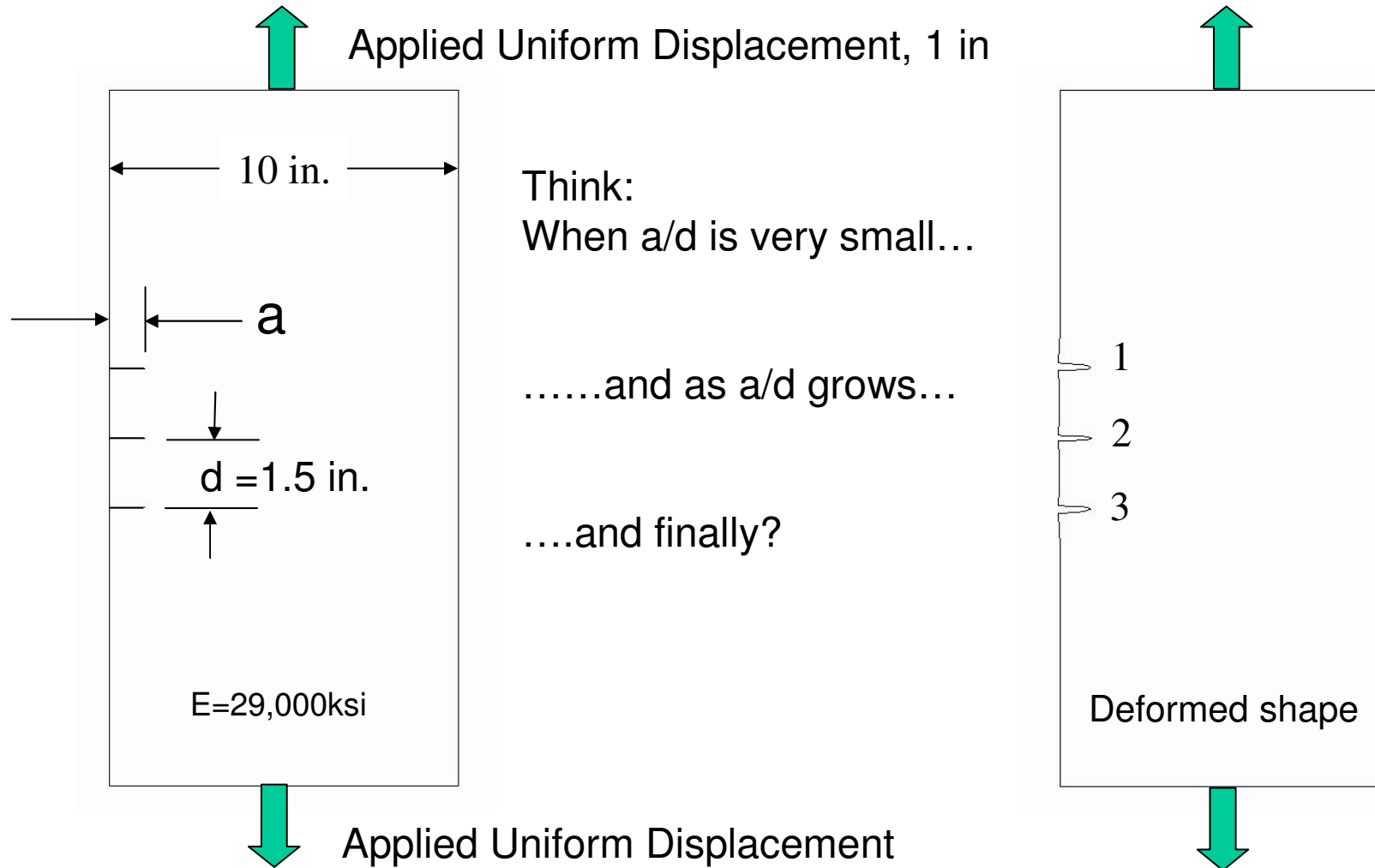
$$(K_I)_i = \sqrt{G_i H}$$

$$\frac{\delta(K_I)_i}{\delta a_j} = \frac{1}{2} \frac{\delta G_i}{\delta a_j} \sqrt{\frac{H}{G_i}} \quad (81)$$

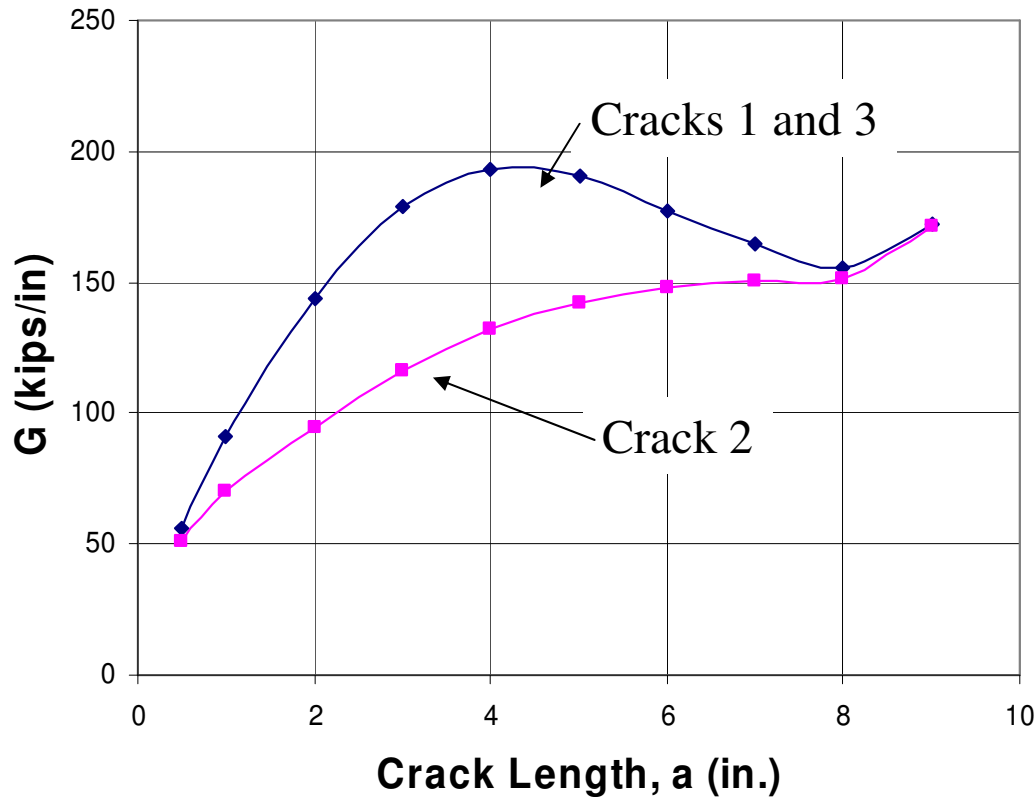
$$\frac{\delta^2(K_I)_i}{\delta a_i^2} = \frac{1}{2} \frac{\delta^2 G_i}{\delta a_i^2} \sqrt{\frac{H}{G_i}} - \frac{1}{4G_i} \left(\frac{\delta G_i}{\delta a_i} \right)^2 \sqrt{\frac{H}{G_i}} \quad (82)$$

Example: Stability of Multiple 2D Crack Systems

Key issue: a/d ratio and its effect on crack tip interaction



Computed Energy Release Rates and Rates of Energy Release Rates



Observe: one can use the generalized VCE technique to predict sensitivity coefficients, strength of crack interaction, and propagation stability of multiple 2D crack systems.

Total Rates of G

a	1, 3	2
0.5	85.2	62.2
1	54.4	19.3
2	39.7	21.4
3	18	11
4	-3.5	10
5	-15	1
6	-33	-5
7	-19	-2
8	-66	-10
9	260	300

Using Eq. 72 and 73

Computed Rates of Energy Release Rates
Using Extended Virtual Crack Extension (EVCE) and Derivative
Technique in FRANC2D

Example Table
of Derivatives
for $a = 4$ in.

i\j	1	2	3	SUM
1	169.6	-101.3	-71.8	-3.5
2	-101.3	212.7	-101.4	10
3	-71.8	-101.4	172.2	-1

$i \neq j$

$$\frac{\delta G_i}{\delta a_j} = -u^T \frac{\delta K}{\delta a_i} K^{-1} \left(\frac{\delta f}{\delta a_j} - \frac{\delta K}{\delta a_j} u \right) + \left(\frac{\delta f}{\delta a_j} - \frac{\delta K}{\delta a_j} u \right)^T K^{-1T} \frac{\delta f}{\delta a_i}$$

$i = j$

$$\frac{\delta G_i}{\delta a_i} = -u^T \frac{\delta K}{\delta a_i} K^{-1} \left(\frac{\delta f}{\delta a_i} - \frac{\delta K}{\delta a_i} u \right) - \frac{1}{2} u^T \frac{\delta^2 K}{\delta a_i^2} u + \left(\frac{\delta f}{\delta a_i} - \frac{\delta K}{\delta a_i} u \right)^T K^{-1T} \frac{\delta f}{\delta a_i} + u^T \frac{\delta^2 f}{\delta a_i^2}$$

2D, Mixed-Mode Case

As previously discussed, we can decompose energy release rate into components from symmetric and anti-symmetric fields (p.86-88)

$$G_i = (G_I)_i + (G_{II})_i$$

$$(G_I)_i = -\frac{1}{2}(u_I)^T \frac{\delta K}{\delta a_i} u_I + (u_I)^T \frac{\delta f_I}{\delta a_i} \quad (G_{II})_i = -\frac{1}{2}(u_{II})^T \frac{\delta K}{\delta a_i} u_{II} + (u_{II})^T \frac{\delta f_{II}}{\delta a_i} \quad (83)$$

Then,

For $i \neq j$

$$\begin{aligned} \frac{\delta(G_I)_i}{\delta a_j} &= -(u_I)^T \frac{\delta K}{\delta a_i} K^{-1} \left(\frac{\delta f_I}{\delta a_j} - \frac{\delta K}{\delta a_j} u_I \right) + \left(\frac{\delta f_I}{\delta a_j} - \frac{\delta K}{\delta a_j} u_I \right)^T K^{-1T} \frac{\delta f_I}{\delta a_i} \\ \frac{\delta(G_{II})_i}{\delta a_j} &= -(u_{II})^T \frac{\delta K}{\delta a_i} K^{-1} \left(\frac{\delta f_{II}}{\delta a_j} - \frac{\delta K}{\delta a_j} u_{II} \right) + \left(\frac{\delta f_{II}}{\delta a_j} - \frac{\delta K}{\delta a_j} u_{II} \right)^T K^{-1T} \frac{\delta f_{II}}{\delta a_i} \end{aligned} \quad (84)$$

2D, Mixed-Mode Case

For $i = j$

$$\begin{aligned} \frac{\delta(G_I)_i}{\delta a_i} = & -(u_I)^T \frac{\delta K}{\delta a_i} K^{-1} \left(\frac{\delta f_I}{\delta a_i} - \frac{\delta K}{\delta a_i} u_I \right) - \frac{1}{2} (u_I)^T \frac{\delta^2 K}{\delta a_i^2} u_I \\ & + \left(\frac{\delta f_I}{\delta a_i} - \frac{\delta K}{\delta a_i} u_I \right)^T K^{-1T} \frac{\delta f_I}{\delta a_i} + u_I^T \frac{\delta^2 f_I}{\delta a_i^2} \end{aligned} \quad (85)$$

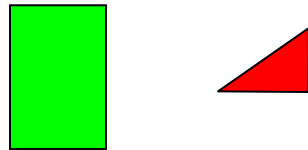
$$\begin{aligned} \frac{\delta(G_{II})_i}{\delta a_i} = & -(u_{II})^T \frac{\delta K}{\delta a_i} K^{-1} \left(\frac{\delta f_{II}}{\delta a_i} - \frac{\delta K}{\delta a_i} u_{II} \right) - \frac{1}{2} (u_{II})^T \frac{\delta^2 K}{\delta a_i^2} u_{II} \\ & + \left(\frac{\delta f_{II}}{\delta a_i} - \frac{\delta K}{\delta a_i} u_{II} \right)^T K^{-1T} \frac{\delta f_{II}}{\delta a_i} + u_{II}^T \frac{\delta^2 f_{II}}{\delta a_i^2} \end{aligned} \quad (86)$$

Crack Growth Model: Principle of Minimum Potential Energy

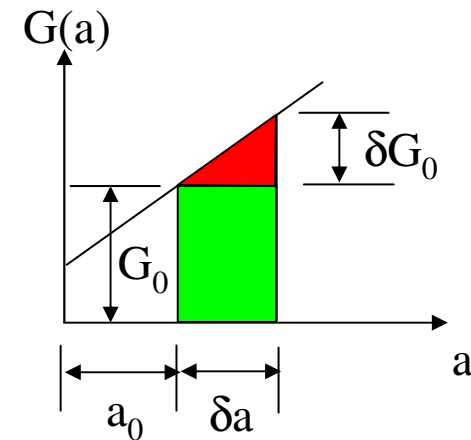
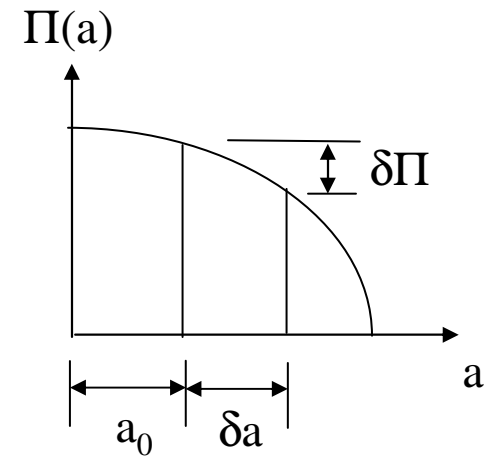


Maximize $-\delta\Pi = \Pi(a_0) - \Pi(a_0 + \delta a)$
w.r.t. Next Crack Extension δa

$$-\delta\Pi = G_0 \delta a + 1/2 \delta G_0 \delta a$$



WE NEED TO KNOW G and dG
for Multiple Planar/Non-Planar Crack Systems
in 2D/3D Under Arbitrary Loading

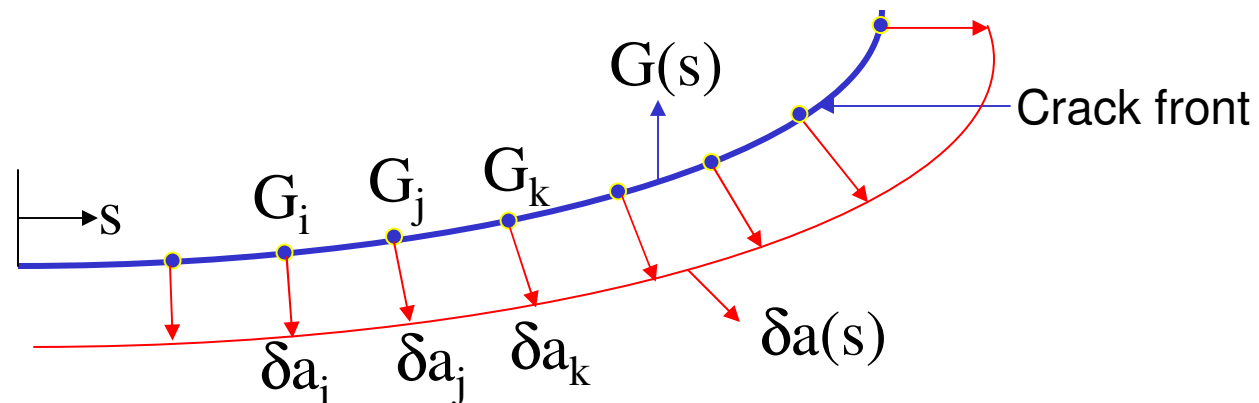


Three Dimensional Problem: Find New Crack Front Shape, $a(s)$

$$\text{Maximize } -\delta\Pi = \int_{\text{Crack Front}} \left[G(s) \delta a(s) + \frac{1}{2} \delta G(s) \delta a(s) \right] ds$$

w.r.t $\delta a(s)$ subjected to
$$\Delta A = \int_{\text{Crack Front}} \delta a(s) ds \quad (87)$$

Piecewise Linearly Approximate $G(s)$, $\delta a(s)$, $\delta G(s)$ along the crack front.



Three Dimensional Problem:

Find New Crack Front Shape, $a(s)$

This problem is ***still*** at the current state-of-the-art. For some interesting insights and examples, see

Hwang, Wawrzynek, Ingrassia. On the virtual crack extension method for calculating the derivatives of energy release rates for a 3D planar crack of arbitrary shape under mode-I loading, *Eng. Fract. Mech.*, **68**, 925-947, 2001.

Hwang C G, Ingrassia A R. Shape prediction and stability analysis of Mode-I planar cracks. *Eng. Fract. Mech.*, **71**, 1751-1777, 2004.