CEE 770 Meeting 4

Objectives of This Meeting

Learn methods for extracting SIF's from local field information:

- Displacement correlation method (DC)
- Virtual crack extension method (VCE)
- Crack closure integral method (CCI)
- J-Integral method (and its generalization, the M-Integral)

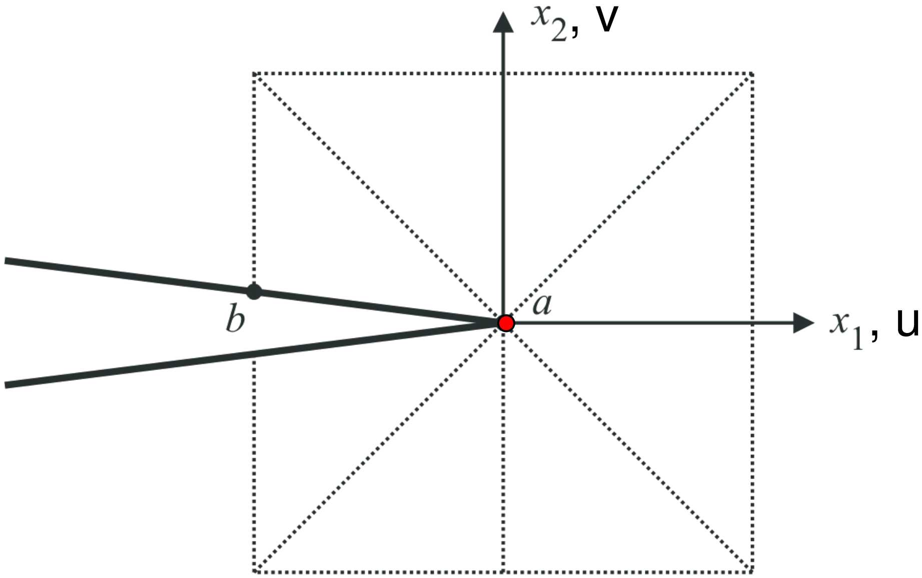
Displacement Correlation Method

The idea is simple:

Correlate computed (FEM/BEM) local displacements with their theoretical values, with SIF as scaling parameter.

- First, recall the form of the theoretical asymptotic displacement fields.
- Next, evaluate these theoretical fields for specific values of displacements at the locations of certain FEM/BEM nodes, with SIF as scaling parameter.
- Next, find FEM/BEM displacements at these nodes.
- Finally, equate the values at these nodes, and solve for the scaling parameter, SIF.

First, recall the form of the theoretical asymptotic displacement fields.



$$u = \frac{K_I}{\mu}\left[\frac{r}{2\pi}\right]^{1/2} \cos\frac{\theta}{2}\left[1-2\nu+\sin^2\frac{\theta}{2}\right] \quad (31)$$

$$v = \frac{K_I}{\mu}\left[\frac{r}{2\pi}\right]^{1/2} \sin\frac{\theta}{2}\left[2-2\nu-\cos^2\frac{\theta}{2}\right] \quad (32)$$

Note: for plane stress, let $\nu = \nu/(1+\nu)$

$$u = \frac{K_{II}}{\mu}\left[\frac{r}{2\pi}\right]^{1/2} \sin\frac{\theta}{2}\left[2-2\nu+\cos^2\frac{\theta}{2}\right] \quad (33)$$

$$v = \frac{K_{II}}{\mu}\left[\frac{r}{2\pi}\right]^{1/2} \cos\frac{\theta}{2}\left[-1+2\nu+\sin^2\frac{\theta}{2}\right] \quad (34)$$

Set r = $r_{a\text{-}b}$, and θ = 180°

$$v_b - v_a = \frac{K_I}{\mu}\left[\frac{r_{a-b}}{2\pi}\right](2-2\nu)$$

$$u_b - u_a = \frac{K_{II}}{\mu}\left[\frac{r_{a-b}}{2\pi}\right](2-2\nu)$$

Displacement Correlation Method

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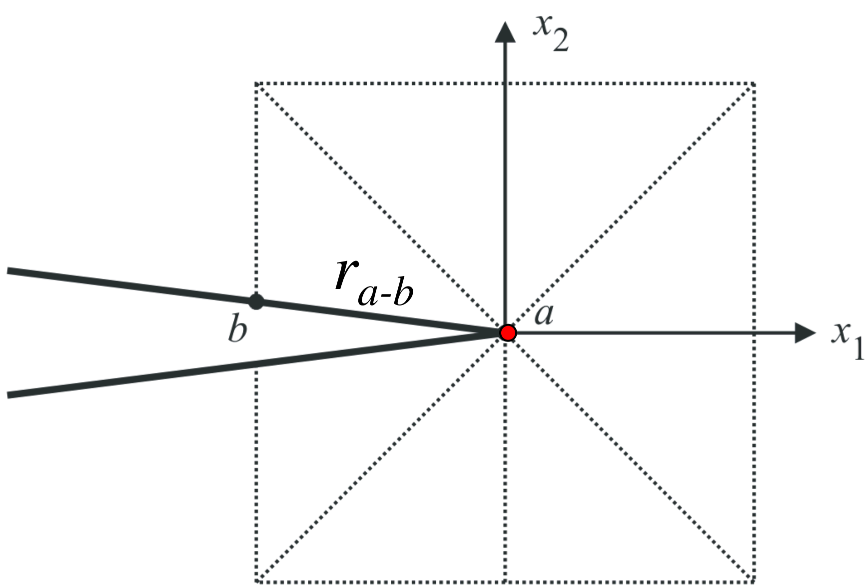
Displacement Correlation Methods with non-Singular Elements

For plane strain case:



$$K_I = \frac{\mu\sqrt{2\pi}\left(v_b - v_a\right)}{\sqrt{r_{a-b}}\left(2-2\nu\right)} \quad (35)$$

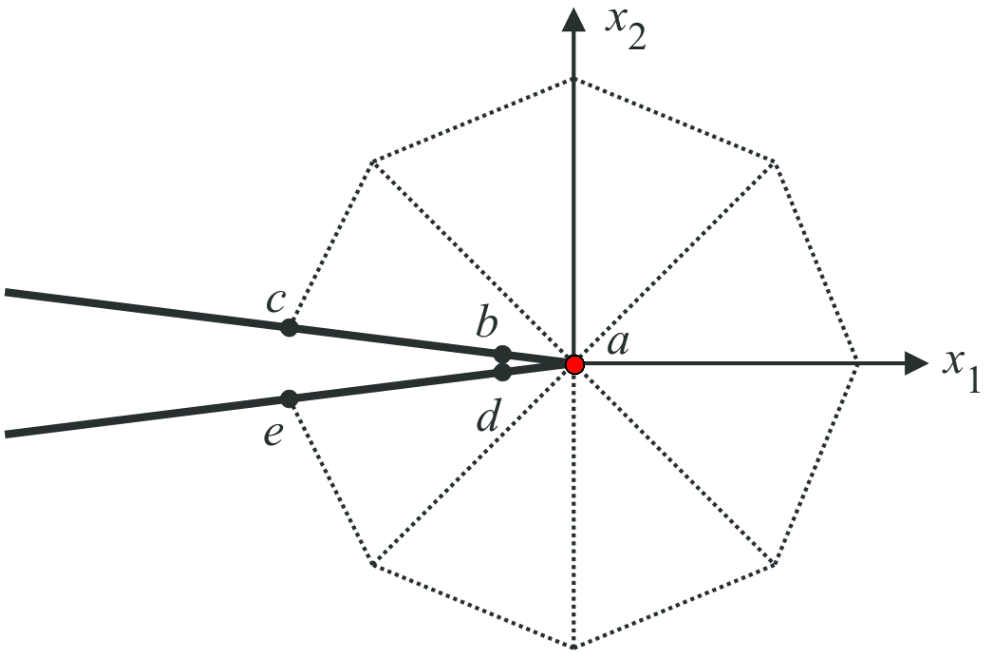
$$K_{II} = \frac{\mu\sqrt{2\pi}\left(u_b - u_a\right)}{\sqrt{r_{a-b}}\left(2-2\nu\right)} \quad (36)$$

$$K_{III} = \frac{\mu\sqrt{\pi}\left(w_b - w_a\right)}{\sqrt{2r_{a-b}}} \quad (37)$$

where μ is the shear modulus, ν is Poisson's ratio, r is the distance from the crack tip to the correlation point, and u_i, v_i, w_i are the *x*, *y*, and *z* displacements at point *i*

The same expressions can be used for plane stress assumptions if ν is replaced with $\nu = \nu / (1+\nu)$.

Displacement Correlation Methods with 1/4-Point Elements: The Approximate Field

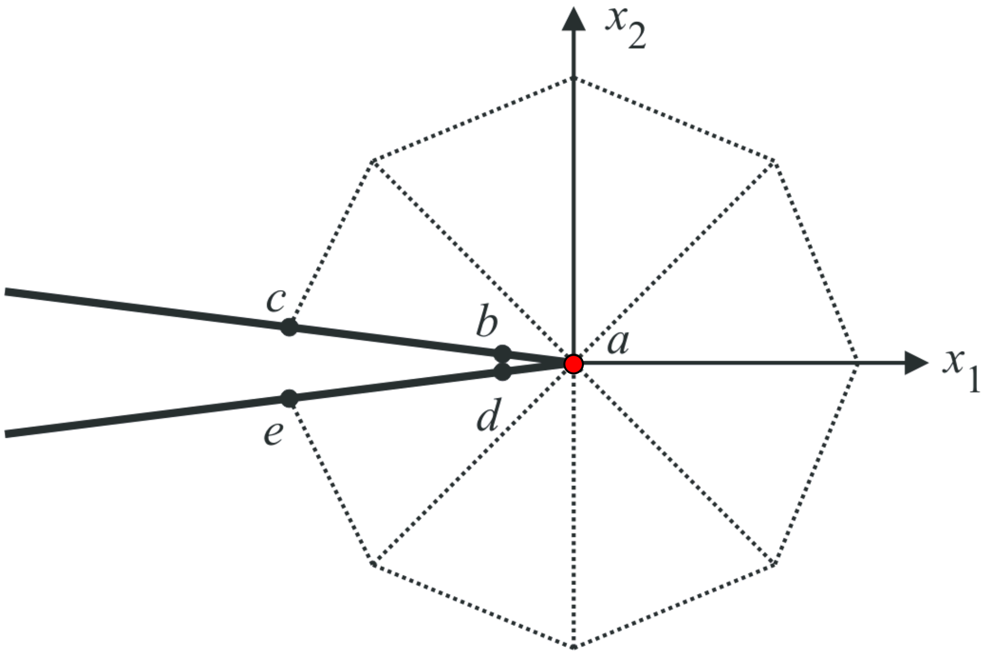


Write the displacement function along $r_{a\text{-}b\text{-}c}$ and along $r_{a\text{-}d\text{-}e}$, using as a model, Equation 30

$$v_{upper} = v_a + (-3v_a + 4v_b - v_c)\sqrt{\frac{r}{l}} + (2v_a - 4v_b + 2v_c)\frac{r}{l} \quad (38)$$

$$v_{lower} = v_a + (-3v_a + 4v_d - v_e)\sqrt{\frac{r}{l}} + (2v_a - 4v_d + 2v_e)\frac{r}{l} \quad (39)$$

Displacement Correlation Methods with 1/4-Point Elements: The Correlation



Evaluate 32 along $\theta = +180°$, $r_{a\text{-}b\text{-}c}$ and -180°, $r_{a\text{-}d\text{-}e}$; find theoretical v_{upper} - v_{lower}.

Equate to approximate value, 40.

Solve for K_I, 41.

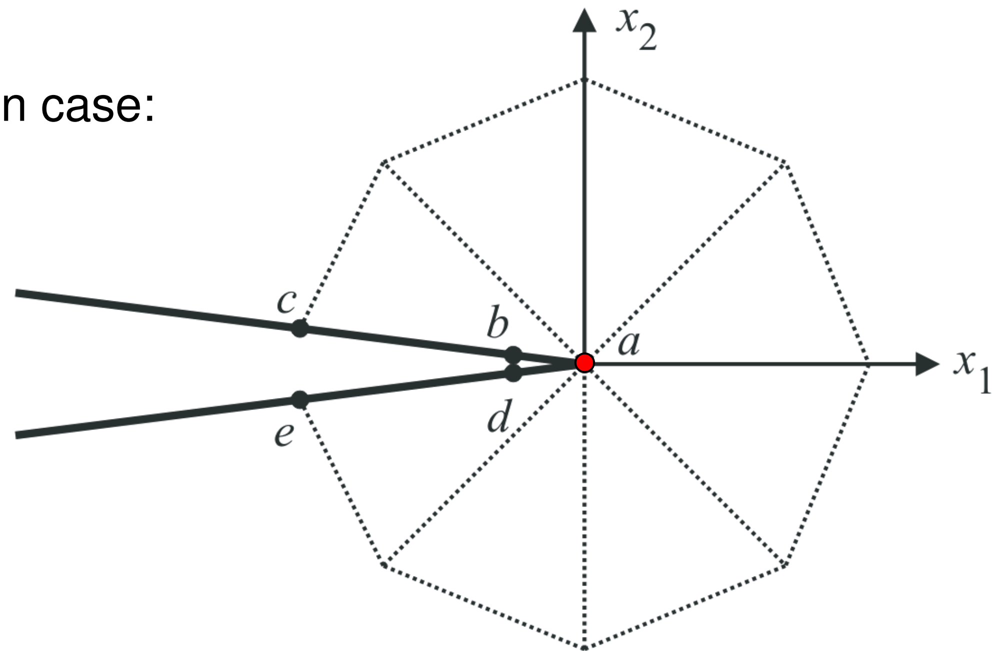
$$v_{upper} - v_{lower} = \left[4(v_b - v_d) + v_e - v_c\right]\sqrt{\frac{r}{l}} + \left[4(v_b - v_d) + 2(v_c - v_e)\right]\frac{r}{l} \qquad (40)$$

$$K_I = \frac{\mu\sqrt{2\pi}}{\sqrt{r_{a-b-c}}\,(2-2\nu)}\left[4(v_b - v_d) + v_e - v_c\right] \qquad (41)$$

For plane strain case. Analogous procedure for K_{II}.

Displacement Correlation Methods with 1/4-Point Elements: The Correlation

For plane strain case:



$$u_{upper} - u_{lower} = \left[4(u_b - u_d) + u_e - u_c\right]\sqrt{\frac{r}{l}} + \left[4(u_b - u_d) + 2(u_c - u_e)\right]\frac{r}{l}$$

$$K_{II} = \frac{\mu\sqrt{2\pi}}{\sqrt{r_{a-b-c}}\,(2-2\nu)}\left[4(u_b - u_d) + u_e - u_c\right] \quad (42)$$

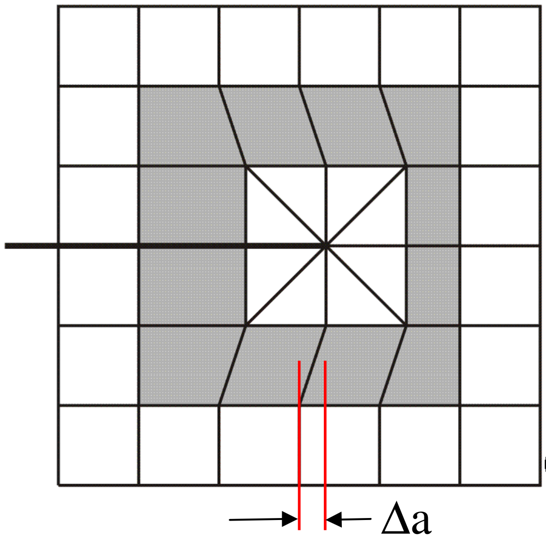
Observations on the Displacement Correlation Method

- Simple, cheap.
- Many variants, eg. correlate with many near-field displacements, then average.
- Relies on local displacement differences, therefore, very dependent on local meshing for accuracy.
- Has analogue: Stress Correlation. Why not use it?

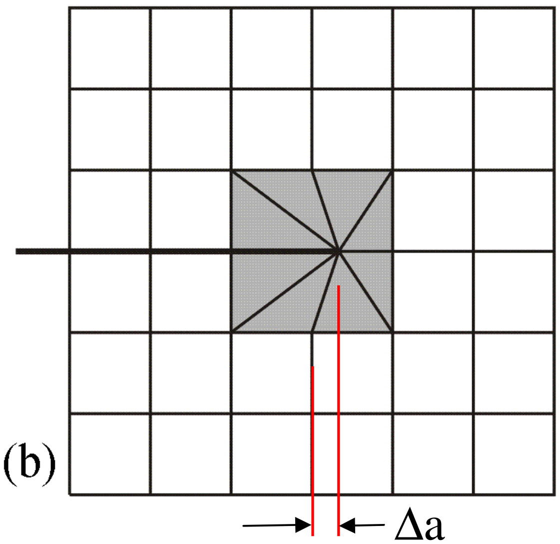
Virtual Crack Extension (VCE) Methods

- The virtual crack extension method is an energy-based approach.
- It computes the rate of change in the total potential energy of a system for a small (virtual) extension of the crack.
- Under LEFM assumptions, this is equal to the energy release rate. This method was first proposed by Parks (1975) and by Hellen (1975).

Virtual Crack Extension (VCE) Methods



(a)

(b)


Two of many possible virtual crack extensions: (a) an annular ring of elements around the crack tip, or (b) just the crack-tip elements. The shaded area indicates elements that have nonzero contributions to equation 45.

Recall from FE theory that the total potential energy of the system, Π, is:

$$\Pi = \tfrac{1}{2}\mathbf{u}^{\mathrm{T}}\boldsymbol{K}\mathbf{u} - \mathbf{u}\boldsymbol{f} \qquad (43)$$

where **u** is the nodal displacement vector, $\boldsymbol{K}$ is the stiffness matrix and $\boldsymbol{f}$ is the external force vector.

Virtual Crack Extension Methods

$$\Pi = \tfrac{1}{2}\mathbf{u}^{\mathrm{T}}\boldsymbol{K}\mathbf{u} - \mathbf{u}\boldsymbol{f}$$

Use definition of energy release rate, G

$$G \equiv -\frac{\partial \Pi}{\partial a} = -\tfrac{1}{2}\mathrm{u}^{\mathrm{T}}\frac{\partial \boldsymbol{K}}{\partial a}\mathrm{u} + \mathrm{u}^{\mathrm{T}}\underbrace{\frac{\partial \boldsymbol{f}}{\partial a}}_{=0} - \frac{\partial \mathrm{u}^{\mathrm{T}}}{\partial a}\underbrace{\left[\boldsymbol{K}\mathrm{u} - \boldsymbol{f}\right]}_{\equiv 0} \quad (44)$$

This form of VCE is also called the Stiffness Derivative Method

$$G = -\tfrac{1}{2}\mathrm{u}^{\mathrm{T}}\boxed{\frac{\partial \boldsymbol{K}}{\partial a}}\mathrm{u} \quad (45)$$

Virtual Crack Extension Methods

Parks and many others compute the stiffness derivatives using a finite difference approach:

$$\frac{\partial \mathbf{K}}{\partial a} \approx \frac{\mathbf{K}_{a+\Delta a} - \mathbf{K}_{a}}{\Delta a} \qquad (46)$$

This approach is simple, but introduces approximation error and the need to select a value for Δa. What to use?????

Haber (1985) substituted an analytical treatment for $\partial \mathbf{K}/\partial a$, which substantially improves the fidelity of the method. Lin and Abel (1988) improved the analytical derivation process, and Hwang, Ingraffea *et al*. (1998) generalized the method to higher order derivatives and 3D.

We will study this latter approach extensively later in this course.

Observations on the Virtual Crack Extension Method

- Not as simple as Displacement Correlation.
- Energy-based, and involves local integrals.
- In general, for the same mesh, expected to produce more accurate SIF’s. Why? Let’s check for the Griffith Problem.

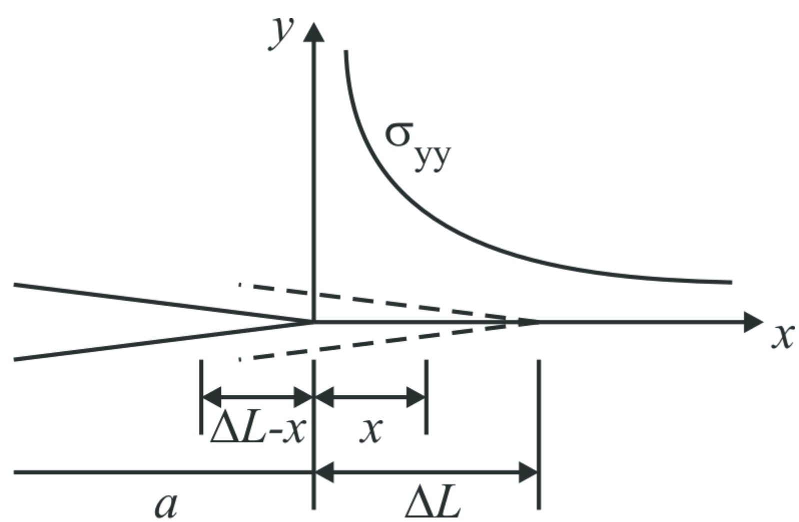
Crack Closure Integral Methods (CCI)

The crack closure approach was first suggested by Rybicki and Kanninen (1977). It is based on Irwin's notion (1957) of reversing crack growth to compute energy release rate.

Irwin's crack closure integrals:

$$G_I = \lim_{\Delta L \to 0} \frac{1}{2\Delta L} \int_0^{\Delta L} \sigma_{yy}(r = x, \theta = 0)\, \mathrm{v}(r = \Delta L - x, \theta = \pi)\, dr \quad (47)$$

$$G_{II} = \lim_{\Delta L \to 0} \frac{1}{2\Delta L} \int_0^{\Delta L} \tau_{xy}(r = x, \theta = 0)\, \mathrm{u}(r = \Delta L - x, \theta = \pi)\, dr \quad (48)$$



Using the FEM for CCI

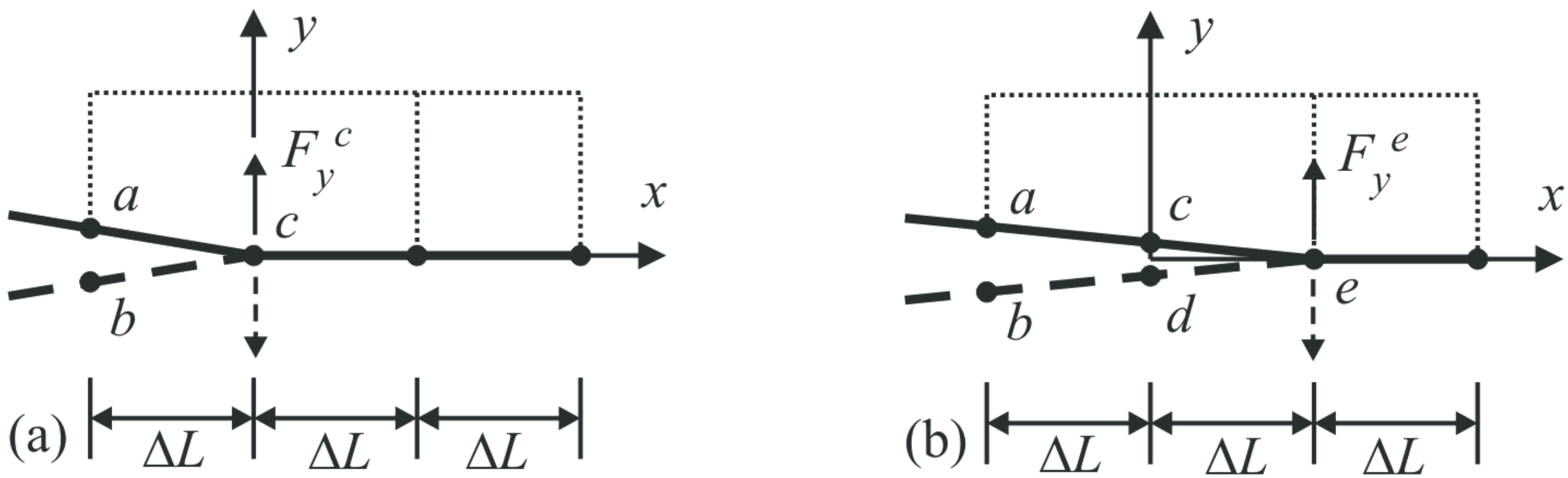
The idea is to:

1. Use the FEM to compute equivalent nodal forces and their conjugate nodal displacements in elements around the crack front. Nodal displacements are primary variables in the FEM, and equivalent nodal forces will be computed more accurately than crack front stresses (why?). Then,

2. Replace the continuous energy integral with a simple, discrete, nodal equivalent work calculation.

Early use of this approach required 2 separate FE calculations, the first in the current crack geometry, the second with a small crack extension (how far, and do we really need 2 analyses?).

Crack Closure Integral Method: Linear Elements



Local mesh configuration used for the MCCI technique: a) first analysis, b) second analysis after the crack has been extended.

$$G_I = \frac{1}{2\Delta L} F_y^c \left(v^c - v^d \right) \quad (49)$$

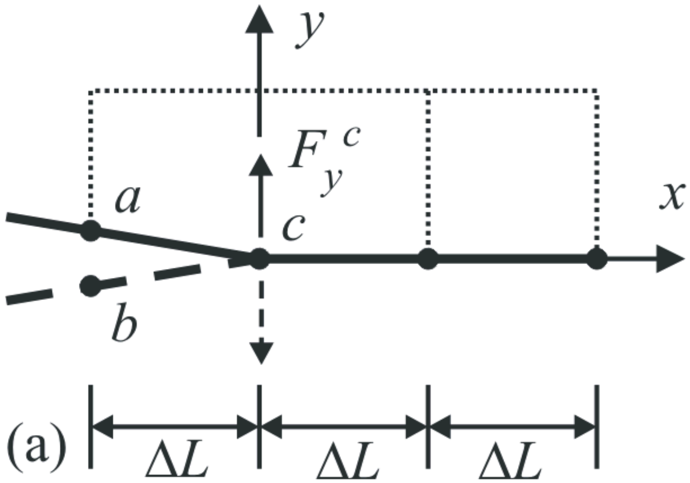
$$G_{II} = \frac{1}{2\Delta L} F_x^c \left(u^c - u^d \right) \quad (50)$$

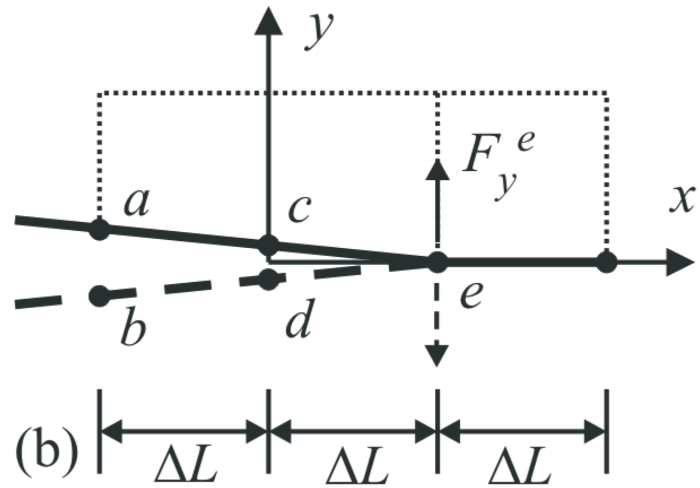
for plane strain

$$K_I = \sqrt{G_I E / (1 - \nu^2)}$$

$$K_{II} = \sqrt{G_{II} E / (1 - \nu^2)}$$

Modified Crack Closure Method: Linear Elements





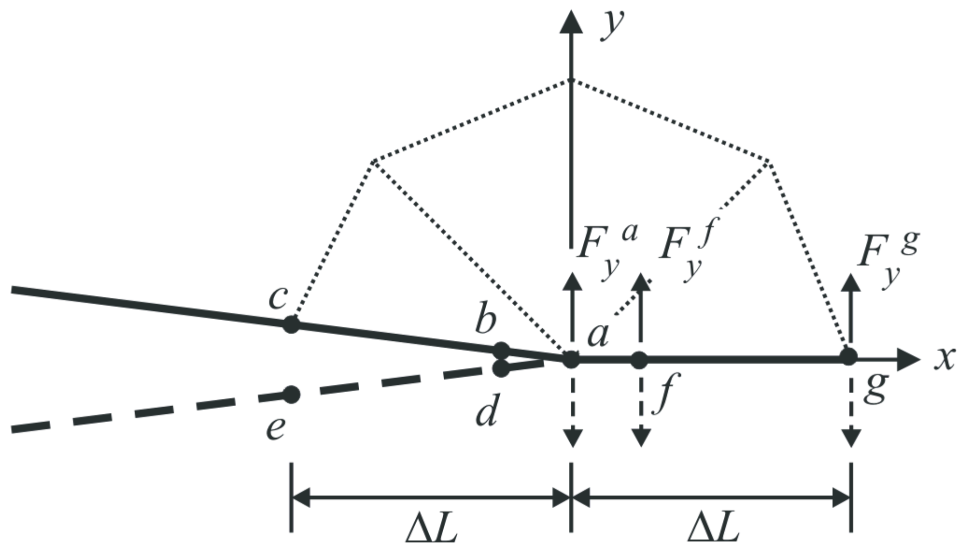
If ΔL is sufficiently short, then $v^c \approx v^a$ and $v^d \approx v^b$
only one FE analysis is needed, and

$$G_I = \frac{1}{2\Delta L} F_y^c \left(v^a - v^b \right) \quad (51)$$

$$G_{II} = \frac{1}{2\Delta L} F_x^c \left(u^a - u^d \right) \quad (52)$$

This is the MCCI method.

MCCI:¼-Point Elements



(51)

$$G_I = \frac{1}{\Delta L}\left[\left(C_{11}F_y^a + C_{12}F_y^f + C_{13}F_y^g\right)\left(\mathrm{v}^b - \mathrm{v}^e\right) + \left(C_{21}F_y^a + C_{22}F_y^f + C_{23}F_y^g\right)\left(\mathrm{v}^c - \mathrm{v}^d\right)\right]$$

$$G_{II} = \frac{1}{\Delta L}\left[\left(C_{11}F_x^a + C_{12}F_x^f + C_{13}F_x^g\right)\left(\mathrm{u}^b - \mathrm{u}^e\right) + \left(C_{21}F_x^a + C_{22}F_x^f + C_{23}F_x^g\right)\left(\mathrm{u}^c - \mathrm{u}^d\right)\right]$$

(52)

$$C_{11} = \frac{33\pi}{2} - 52, \quad C_{12} = 17 - \frac{21\pi}{4}, \quad C_{13} = \frac{21\pi}{2} - 32$$

$$C_{21} = 14 - \frac{33\pi}{8}, \quad C_{22} = \frac{21\pi}{6} - \frac{7}{2}, \quad C_{23} = 8 - \frac{21\pi}{8}$$

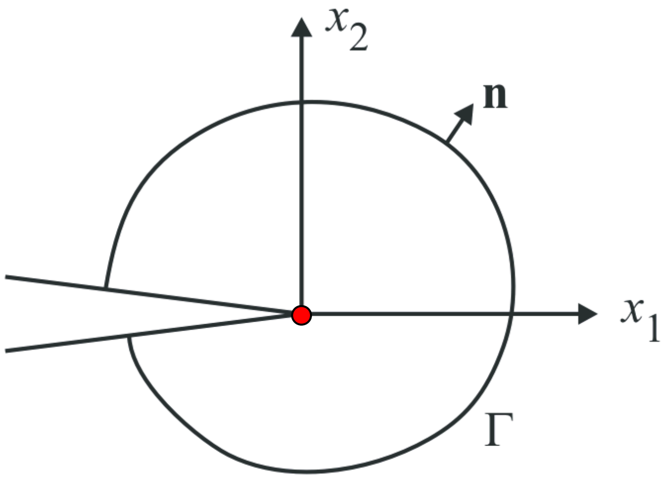
Observations on the MCCI

1. In general, for the same mesh, the MCCI method should produce more accurate SIF's than Displacement Correlation. Why? Let's check for the Griffith problem.

2. The MCCI method has been generalized for arbitrary numerical techniques and field interpolations in

> Singh, R., Carter, B., Wawrzynek, P., Ingraffea, A., "Universal Crack Closure Integral for SIF Estimation", *Engineering Fracture Mechanics*, **60**, 2, 133-146, 1998.

The J-Integral (2-D): Contour Version



J-integral is path-independent if:
- no body forces inside the integration area,
- no tractions on the crack surface, and
- material behavior is elastic (linear or nonlinear).

Path independence for cases with body forces or crack-face tractions require additional terms in the integral.

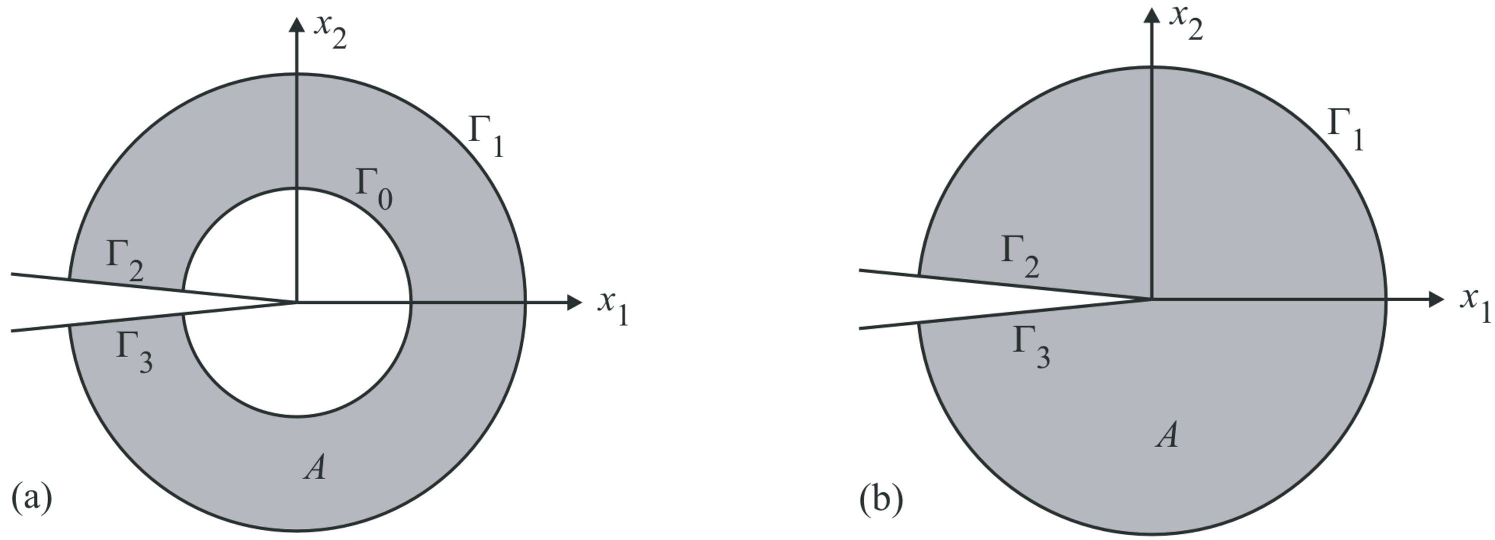
Rice, 1968

$$J \equiv \lim_{\Gamma \to 0} \int_{\Gamma} \left[W n_1 - \sigma_{ij} \frac{\partial u_i}{\partial x_1} n_j \right] d\Gamma \qquad (53)$$

where W is the strain energy density, **σ** is the stress tensor, **n** is the unit outward normal to the contour, and **u** is the displacement vector (summation convention used over identical indices)

Under linear elastic material assumptions, the *J*-integral can be interpreted as being equivalent to the energy release rate, *G*.

The J-Integral (2-D): Area Version



$$\bar{J} = \int_A \left[\sigma_{ij} \frac{\partial u_i}{\partial x_1} - W\delta_{1j} \right] \frac{\partial q_1}{\partial x_j} \, dA \qquad (54)$$

where δ is the Kronecker delta and q is a weighting function defined over the domain of integration. Physically, q can be thought of as the displacement field due to a virtual crack extension.

The J-Integral (2-D): Area Version

The q function is defined by prescribing nodal values that are interpolated over elements in the domain using the standard shape functions:

$$q = \sum_i N_i q_i \qquad \text{and} \qquad \frac{\partial q}{\partial x_j} = \sum_i \frac{\partial N_i}{\partial x_j} q_i$$

The other quantities in equation 54 are easily computed in a finite element context (eg., $W = ½\sigma_{ij}\varepsilon_{ij}$).

The J-Integral (2-D): Area Version with Crack Face Tractions

If there are tractions on the crack faces, an additional term must be added to the *J*-integral. For crack face tractions t_i this is

$$\bar{J} = \bar{J}_A + \bar{J}_\Gamma = \bar{J}_A + \int_{\Gamma_2+\Gamma_3} t_i \frac{\partial u_i}{\partial x_1} q \, d\Gamma \qquad (55)$$

where J_A is given by equation 54.

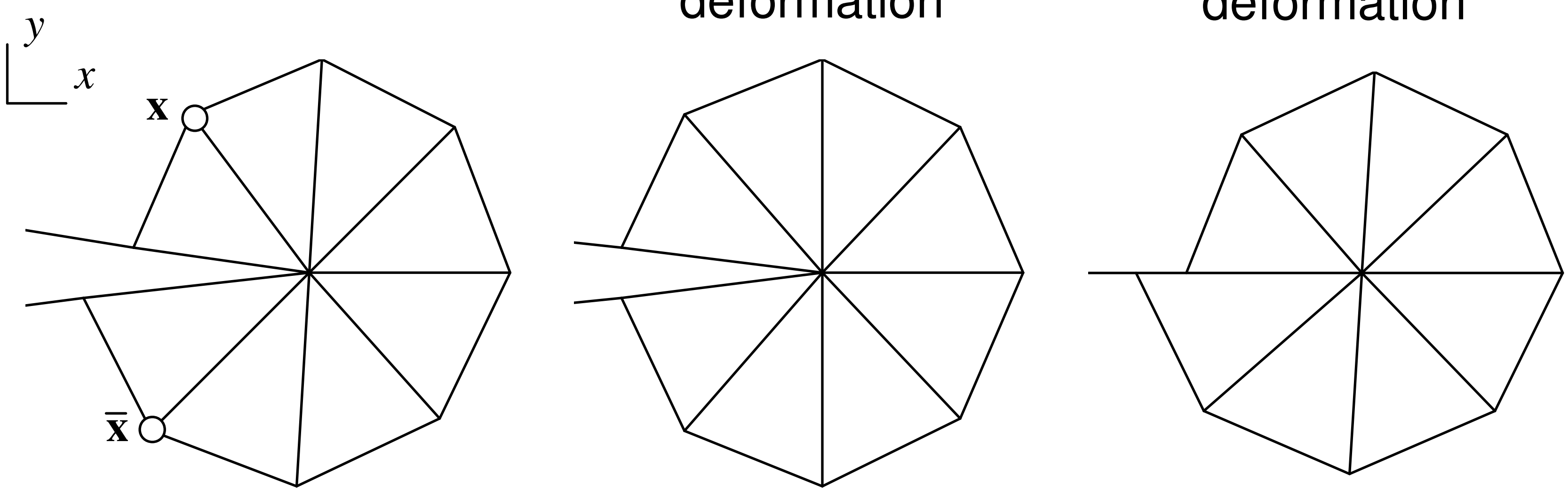
How to Extract Mixed-Mode SIF's from the J-Integral?

$$\bar{J} = G = \left(K_I^2 + K_{II}^2\right)/E \qquad \text{(plane stress)} \qquad (56)$$

Separate the modes by decomposing the near crack-tip displacement fields into one field that is symmetric with respect to the crack and another field that is anti-symmetric with respect to the crack.

Decomposition of Displacement Fields for Mode Separation

total deformation | symmetric deformation | anti-symmetric deformation



$$\mathsf{u} = \mathsf{u}_{sym} + \mathsf{u}_{anti-sym}$$

$$\mathsf{u}_{sym} = \tfrac{1}{2}\begin{Bmatrix} \mathsf{u} + \bar{\mathsf{u}} \\ \mathsf{v} - \bar{\mathsf{v}} \end{Bmatrix}$$

$$\mathsf{u}_{anti-sym} = \tfrac{1}{2}\begin{Bmatrix} \mathsf{u} - \bar{\mathsf{u}} \\ \mathsf{v} + \bar{\mathsf{v}} \end{Bmatrix}$$

Decomposition of Stress Fields for Mode Separation

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{sym} + \boldsymbol{\sigma}_{anti-sym} = \tfrac{1}{2}\begin{bmatrix} \sigma_{11}+\bar{\sigma}_{11} & \sigma_{12}-\bar{\sigma}_{12} \\ sym & \sigma_{22}+\bar{\sigma}_{22} \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} \sigma_{11}-\bar{\sigma}_{11} & \sigma_{12}+\bar{\sigma}_{12} \\ sym & \sigma_{22}-\bar{\sigma}_{22} \end{bmatrix}$$

Then, use equation 54 to evaluate:

$$G_I = J_I = J(\mathrm{u}_{sym}, \boldsymbol{\sigma}_{sym})$$

$$G_{II} = J_{II} = J(\mathrm{u}_{anti-sym}, \boldsymbol{\sigma}_{anti-sym}) \qquad (57)$$

$$K_I = \sqrt{\frac{EG_I}{\kappa}}$$

$$K_{II} = \sqrt{\frac{EG_{II}}{\kappa}}$$

$$\kappa = \begin{cases} 1 & \text{plane stress} \\ 1-\nu^2 & \text{plane strain} \end{cases}$$

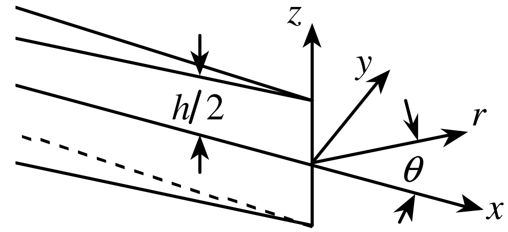
note:

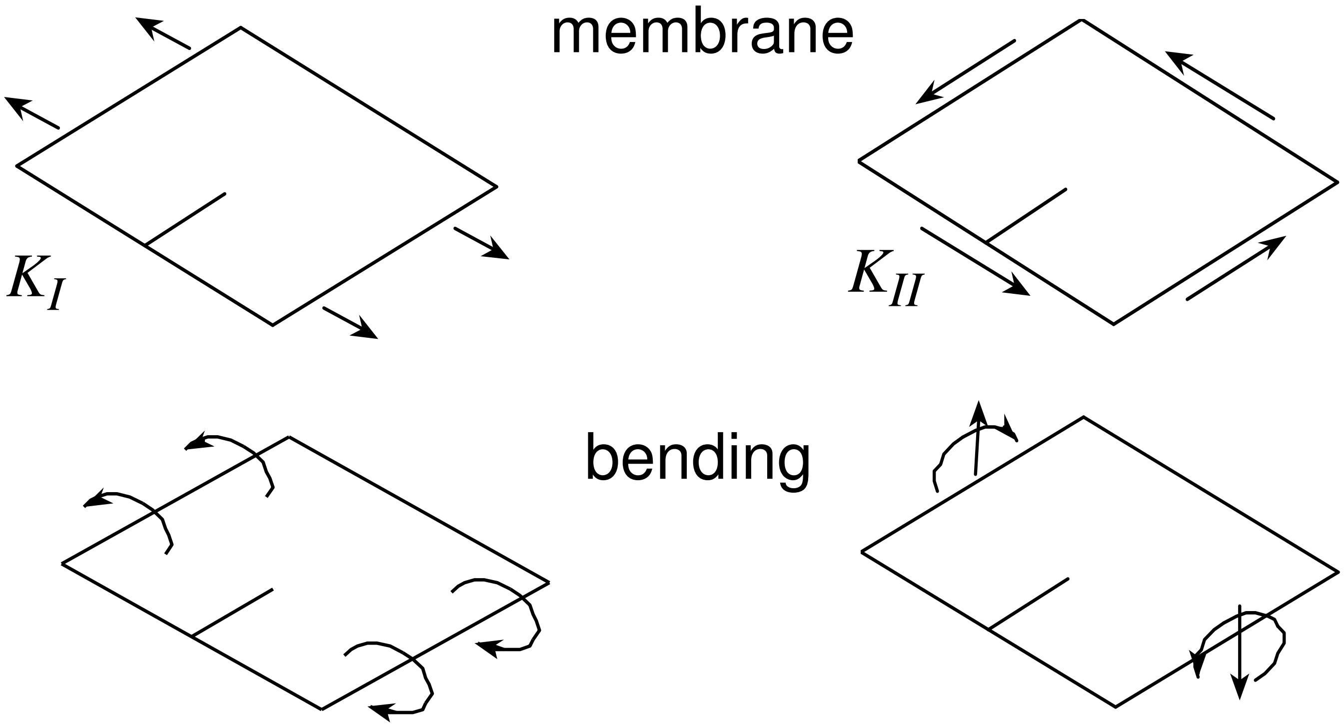
$$J_I, J_{II} \neq J_1, J_2$$

$$J_1 = K_I^2 + K_{II}^2$$

$$J_2 = K_I K_{II}$$

The four shell fracture modes





K_1	Reissner theory	K_2, K_3	Reissner theory
k_1	Kirchhoff theory	k_2	Kirchhoff theory

Near-tip stress fields for Kirchhoff plate theory

$$\begin{Bmatrix} \sigma_r \\ \sigma_{r\theta} \\ \sigma_\theta \end{Bmatrix} = \frac{k_1}{(3+\nu)\sqrt{2r}} \frac{z}{2h} \begin{Bmatrix} (3+5\nu)\cos(\theta/2) - (7+\nu)\cos(3\theta/2) \\ -(1-\nu)\sin(\theta/2) + (7+\nu)\sin(3\theta/2) \\ (5+3\nu)\cos(\theta/2) - (7+\nu)\cos(3\theta/2) \end{Bmatrix}$$

$$+ \frac{k_2}{(3+\nu)\sqrt{2r}} \frac{z}{2h} \begin{Bmatrix} (3-5\nu)\sin(\theta/2) + (5+3\nu)\sin(3\theta/2) \\ -(1-\nu)\cos(\theta/2) + (5+3\nu)\cos(3\theta/2) \\ -2(5+3\nu)\cos(\theta/2)\sin(\theta) \end{Bmatrix} \qquad (58)$$

$$\begin{Bmatrix} \sigma_{rz} \\ \sigma_{\theta z} \end{Bmatrix} = \frac{\left[1 - \left(\frac{2z}{h}\right)^2\right] h/2}{(3+\nu)(2r)^{3/2}} \begin{Bmatrix} -k_1\cos(\theta/2) + k_2\sin(\theta/2) \\ -k_1\sin(\theta/2) - k_2\cos(\theta/2) \end{Bmatrix} \qquad \sigma_z = 0$$

The $r^{3/2}$ term in shear stresses is because the traction-free conditions on the crack faces cannot be satisfied fully with the Kirchhoff assumptions.

The region of dominance of these crack-tip fields is approximately $L/10$, where L is the crack length.

Near-tip stress fields for Reissner plate theory

$$\begin{Bmatrix} \sigma_r \\ \sigma_{r\theta} \\ \sigma_\theta \end{Bmatrix} = \frac{K_1}{\sqrt{2r}} \frac{z}{2h} \begin{Bmatrix} 5\cos(\theta/2) - \cos(3\theta/2) \\ \sin(\theta/2) + \sin(3\theta/2) \\ 3\cos(\theta/2) + \cos(3\theta/2) \end{Bmatrix} + \frac{K_2}{\sqrt{2r}} \frac{z}{2h} \begin{Bmatrix} -5\sin(\theta/2) + 3\sin(3\theta/2) \\ \cos(\theta/2) + 3\cos(3\theta/2) \\ -3\sin(\theta/2) - 3\sin(3\theta/2) \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{rz} \\ \sigma_{\theta z} \end{Bmatrix} = \frac{K_3}{\sqrt{2r}} \left[1 - \left(\frac{2z}{h} \right)^2 \right] \begin{Bmatrix} \sin(\theta/2) \\ \cos(\theta/2) \end{Bmatrix} \qquad \sigma_z = 0 \qquad (59)$$

All stress components have a $r^{-1/2}$ singularity.

The region of dominance of these crack-tip fields is approximately $h/10$, where h is the plate thickness.

Hui and Zehnder have shown that the Kirchhoff and Reissner stress-intensity factors are related

$$k_1 = a_1 K_1 \qquad\qquad k_2 = a_2 K_2 + a_3 K_3$$

with $K_1 / k_1 = [(1+\nu)/(3+\nu)]^{1/2}$ and $k_2^2 \dfrac{1+\nu}{3+\nu} = K_2^2 + K_3^2 \dfrac{8(1+\nu)}{5}$

also $G_I = \dfrac{h K_I^2}{E} \qquad G_{II} = \dfrac{h K_{II}^2}{E} \qquad G_1 = \dfrac{k_1^2 h \pi (1+\nu)}{3E(3+\nu)} \qquad G_2 = \dfrac{k_2^2 h \pi (1+\nu)}{3E(3+\nu)}$

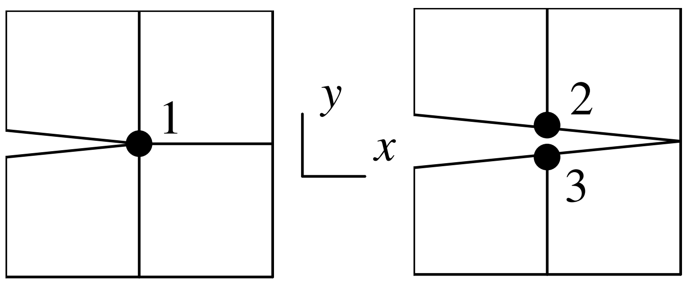
(60)

- This implies that the Reissner fields are controlled by, and can be characterized by the surrounding Kirchhoff fields.
- Because the region of dominance of the Reissner fields is so small (*h*/10), it will often be smaller than the plastic zone size.
- The Kirchhoff assumptions are most appropriate for large scale FEM analyses.

Hui and Zehnder, "A theory for the fracture of thin plates subjected to bending and twisting moments," *Int. J. Fracture*, Vol. 61 (1993) pp. 211-229.

Extracting stress-intensity factors from shell FEM results using Irwin's crack closure integral

method 1: (two analyses)



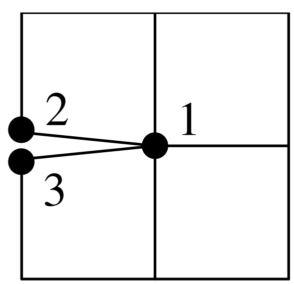
$$G_I = \tfrac{1}{2}\left[F_{1y}(\mathrm{u}_{2y} - \mathrm{u}_{3y}) + M_{1z}(\theta_{2z} - \theta_{3z})\right]$$

$$G_{II} = \tfrac{1}{2}\left[F_{1x}(\mathrm{u}_{2x} - \mathrm{u}_{3x})\right]$$

$$G_1 = \tfrac{1}{2}\left[M_{1x}(\theta_{2x} - \theta_{3x})\right] \qquad (61)$$

$$G_2 = \tfrac{1}{2}\left[F_{1z}(\mathrm{u}_{2z} - \mathrm{u}_{3z}) + M_{1y}(\theta_{2y} - \theta_{3y})\right]$$

method 2: (one analysis)



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