CEE 770 Meeting 3

Objectives of This Meeting

- Review some basic concepts of finite/boundary element interpolation to give insight into the accuracy of these methods for fracture mechanics calculations.
- Introduce the notion of "singular finite/boundary elements".
- Study one particular form of such elements, the ¼-point formulation.

Recall the Mathematical Form of the LEFM Crack Front Fields

Williams **expansion of crack tip stress field (1957):**

Mode I

$$\sigma_x = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^1 \left[\left(2 + \frac{n}{2} + (-1)^n \right) \cos(\frac{n}{2}-1)\theta - (\frac{n}{2}-1)\cos(\frac{n}{2}-3)\theta \right] \quad (1)$$

$$\sigma_y = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^1 \left[\left(2 - \frac{n}{2} - (-1)^n \right) \cos(\frac{n}{2}-1)\theta + (\frac{n}{2}-1)\cos(\frac{n}{2}-3)\theta \right] \quad (2)$$

$$\tau_{xy} = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^1 \left[(\frac{n}{2}-1)\sin(\frac{n}{2}-3)\theta - \left(\frac{n}{2} + (-1)^n \right) \sin(\frac{n}{2}-1)\theta \right] \quad (3)$$

$$\mathrm{u} = \sum_{n=1}^{\infty} \frac{r^{\frac{n}{2}}}{2\mu} a_n^1 \left[\left(\kappa + \frac{n}{2} + (-1)^n \right) \cos\frac{n}{2}\theta - \frac{n}{2}\cos(\frac{n}{2}-2)\theta \right] \quad (4)$$

$$\mathrm{v} = \sum_{n=1}^{\infty} \frac{r^{\frac{n}{2}}}{2\mu} a_n^1 \left[\left(\kappa - \frac{n}{2} - (-1)^n \right) \sin\frac{n}{2}\theta + \frac{n}{2}\sin(\frac{n}{2}-2)\theta \right] \quad (5)$$

Recall the Mathematical Form of the LEFM Crack Front Fields

***Williams* expansion of crack tip stress field (1957):**

Mode II

$$\sigma_x = -\sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^2 \left[\left(2 + \frac{n}{2} - (-1)^n \right) \sin(\frac{n}{2}-1)\theta - (\frac{n}{2}-1)\sin(\frac{n}{2}-3)\theta \right] \qquad (6)$$

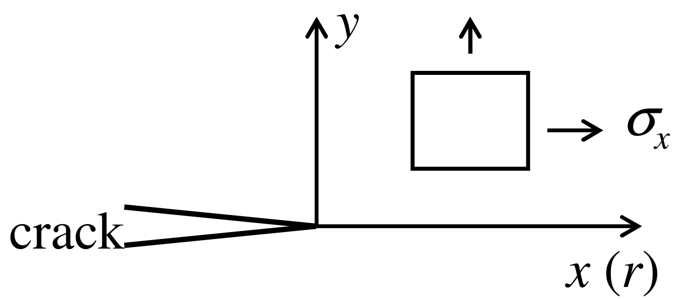
$$\sigma_y = -\sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^2 \left[\left(2 - \frac{n}{2} + (-1)^n \right) \sin(\frac{n}{2}-1)\theta + (\frac{n}{2}-1)\sin(\frac{n}{2}-3)\theta \right] \qquad (7)$$

$$\tau_{xy} = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^2 \left[(\frac{n}{2}-1)\cos(\frac{n}{2}-3)\theta - \left(\frac{n}{2} - (-1)^n \right) \cos(\frac{n}{2}-1)\theta \right] \qquad (8)$$

$$u = -\sum_{n=1}^{\infty} \frac{r^{\frac{n}{2}}}{2\mu} a_n^2 \left[\left(\kappa + \frac{n}{2} - (-1)^n \right) \sin\frac{n}{2}\theta - \frac{n}{2}\sin(\frac{n}{2}-2)\theta \right] \qquad (9)$$

$$v = \sum_{n=1}^{\infty} \frac{r^{\frac{n}{2}}}{2\mu} a_n^2 \left[\left(\kappa - \frac{n}{2} + (-1)^n \right) \cos\frac{n}{2}\theta + \frac{n}{2}\cos(\frac{n}{2}-2)\theta \right] \qquad (10)$$

Example of Expansion Along Crack Line, x = r



$$\sigma_x = \frac{a_1}{\sqrt{r}} + 4a_2 + 3a_3\sqrt{r} + 8a_4 r + 5a_5 r^{3/2} + \cdots \qquad (11)$$

$$\sigma_y = \frac{a_1}{\sqrt{r}} + 3a_3\sqrt{r} + 5a_5 r^{3/2} + \cdots \qquad (12)$$

First (leading) or singular term a_1: *stress intensity factor*

Second term a_2: *T*-stress

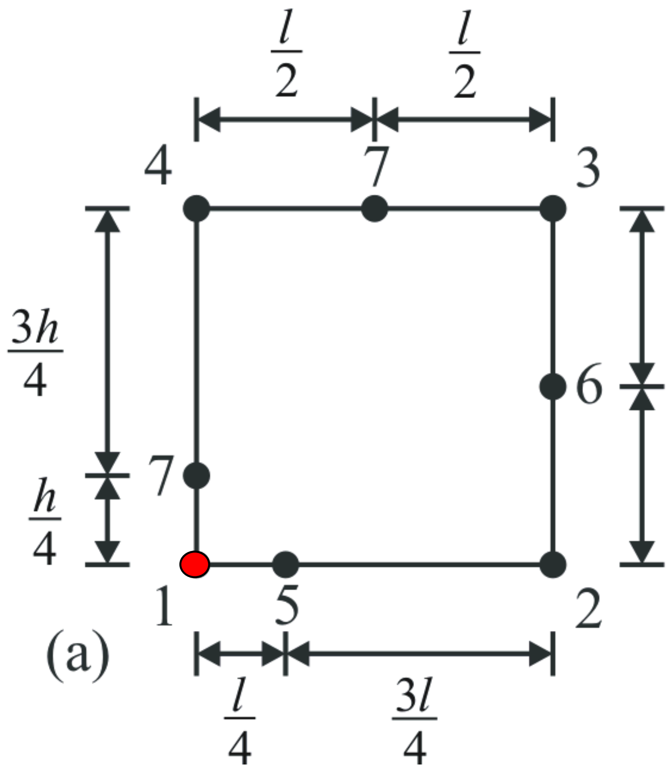
Third term a_3: the leading higher order term

Polynomials versus Crack Front Fields

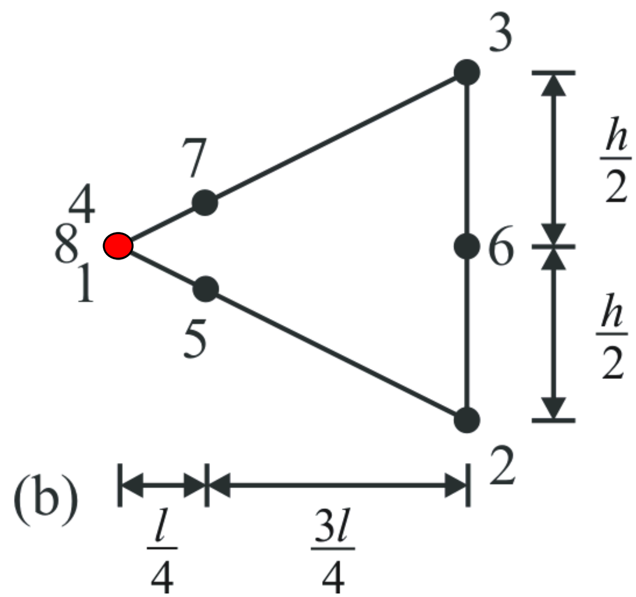
- We saw in the FEM evaluation of the Griffith problem that standard FEM/BEM polynomial-based formulations are a poor match for the nature of LEFM crack front fields.
- If one wants to reproduce these fields with FEM/BEM well, many small dumb (or fewer larger smarter) elements are needed, especially in the K-dominant region. BAD.
- Worse, in the limit as one approaches the crack front, we can never reproduce the singular nature with polynomial-based elements. This makes calculating SIF's accurately using local field information (one of 2 possible approaches) difficult and inefficient with such elements.
- Therefore, many really smart people have invented various forms of "singular finite/boundary elements" (See Chapter 2 of Ingraffea and Wawrzynek).
- Here we will study in detail one such form, the so-called 1/4-point form.

¼-Point Elements: Where Do They Get Their Name?

Henshell and Shaw, 1975



Barsoum, 1976

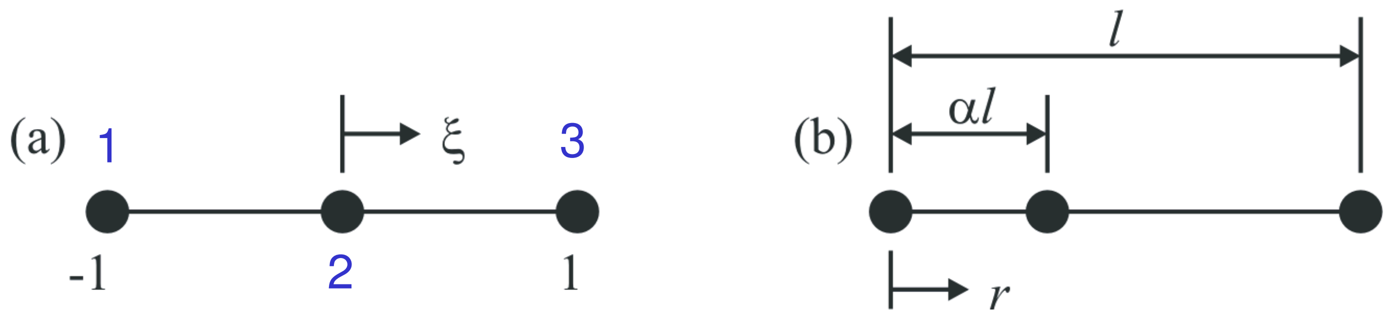


Quadrilateral (a) and collapsed quadrilateral (b) quarter-point elements.

¼-Point Elements: How Do They Work?

- The miss-placing of a mid-side node introduces a singularity into the mapping between the element's parametric coordinate space and Cartesian space.
- We will first demonstrate this using a one-dimensional finite element (same as a 2-dimensional boundary element).
- We will then make observations of good and bad versions of this class of element in 2D and 3D.

1D FEM/2D BEM, ¼-Point Singular Element



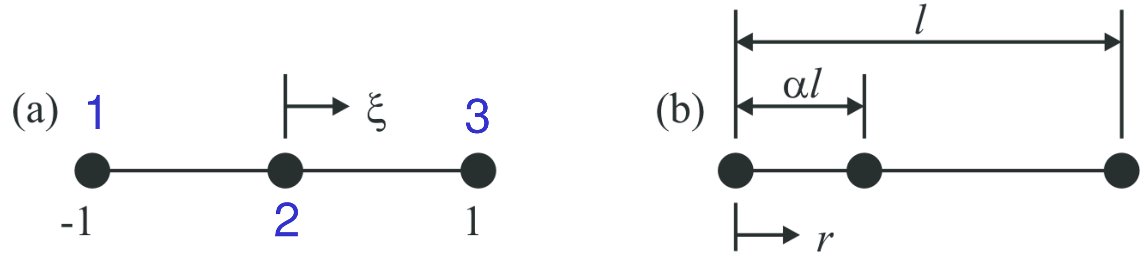
A quadratic element, (a) the parametric space of the element, (b) the Cartesian space of the element. The crack tip is at r=0.

$$\mathrm{u} = \sum_{i=1}^{3} N_i \mathrm{u}_i = \tfrac{1}{2}\xi(\xi-1)\mathrm{u}_1 + (1-\xi^2)\mathrm{u}_2 + \tfrac{1}{2}\xi(\xi+1)\mathrm{u}_3 \qquad (26)$$

Its standard, polynomial *displacement* interpolation scheme, above, regrouped, below.

$$\mathrm{u} = \mathrm{u}_2 + \tfrac{1}{2}(\mathrm{u}_3 - \mathrm{u}_1)\xi + (\tfrac{1}{2}(\mathrm{u}_1 + \mathrm{u}_3) - \mathrm{u}_2)\xi^2 \qquad (27)$$

1D FEM/2D BEM, ¼-Point Singular Element



$$r=\sum_{i=1}^{3}N_i r_i=\alpha l+\tfrac{1}{2}l\xi+l(\tfrac{1}{2}-\alpha)\xi^2 \qquad (28)$$

Its standard, polynomial *geometry* interpolation scheme.

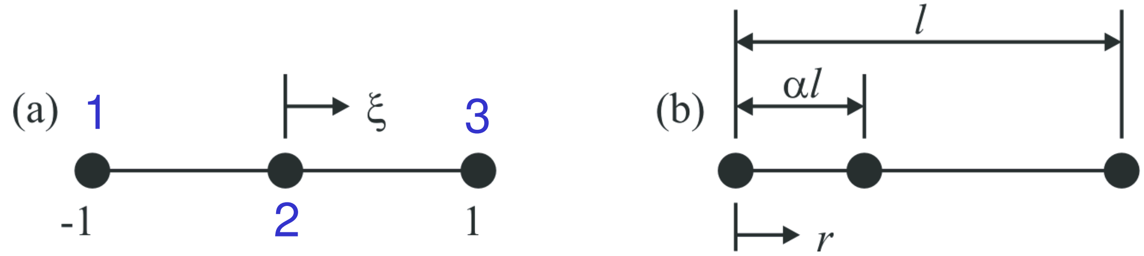
First, the usual case of mid-side geometry:

$$\alpha=\frac{1}{2} \longrightarrow \xi=\frac{2r}{l}-1$$

Then, from (27), the expected, polynomial interpolation:

$$\mathrm{u}=\mathrm{u}_1+(-3\mathrm{u}_1+4\mathrm{u}_2-\mathrm{u}_3)\frac{r}{l}+2(\mathrm{u}_1-2\mathrm{u}_2+\mathrm{u}_3)\frac{r^2}{l^2} \qquad (29)$$

1D FEM/2D BEM, ¼-Point Singular Element



$$r = \sum_{i=1}^{3} N_i r_i = \alpha l + \tfrac{1}{2} l\xi + l(\tfrac{1}{2} - \alpha)\xi^2 \qquad (28)$$

Its standard, polynomial geometry interpolation scheme.

Next, the unusual case of ¼-point geometry:

$$\alpha = \frac{1}{4} \longrightarrow \xi = \frac{2\sqrt{lr}}{l} - 1$$

Then, from (27), the *unexpected*, non-polynomial interpolation !!!:

$$\mathrm{u} = \mathrm{u}_1 + 2(\mathrm{u}_1 - 2\mathrm{u}_2 + \mathrm{u}_3)\frac{r}{l} + (-3\mathrm{u}_1 + 4\mathrm{u}_2 + \mathrm{u}_3)\frac{\sqrt{lr}}{l} \qquad (30)$$

1D FEM/2D BEM, ¼-Point Singular Element

Normal displacement field: $$\mathrm{u} = \mathrm{u}_1 + (-3\mathrm{u}_1 + 4\mathrm{u}_2 - \mathrm{u}_3)\frac{r}{l} + 2(\mathrm{u}_1 - 2\mathrm{u}_2 + \mathrm{u}_3)\frac{r^2}{l^2}$$

¼-Point displacement field: $$\mathrm{u} = \mathrm{u}_1 + 2(\mathrm{u}_1 - 2\mathrm{u}_2 + \mathrm{u}_3)\frac{r}{l} + (-3\mathrm{u}_1 + 4\mathrm{u}_2 + \mathrm{u}_3)\frac{\sqrt{lr}}{l}$$

Compare to theoretical field: $$\mathrm{u} = \sum_{n=1}^{\infty} \frac{r^{\frac{n}{2}}}{2\mu} a_n^1 \left[\left(\kappa + \frac{n}{2} + (-1)^n \right) \cos\frac{n}{2}\theta - \frac{n}{2}\cos(\frac{n}{2} - 2)\theta \right]$$

Normal strain field: $$\varepsilon = \frac{d\mathrm{u}}{dr} = (-3\mathrm{u}_1 + 4\mathrm{u}_2 - \mathrm{u}_3)\frac{1}{l} + 4(\mathrm{u}_1 - 2\mathrm{u}_2 + \mathrm{u}_3)\frac{r}{l^2}$$

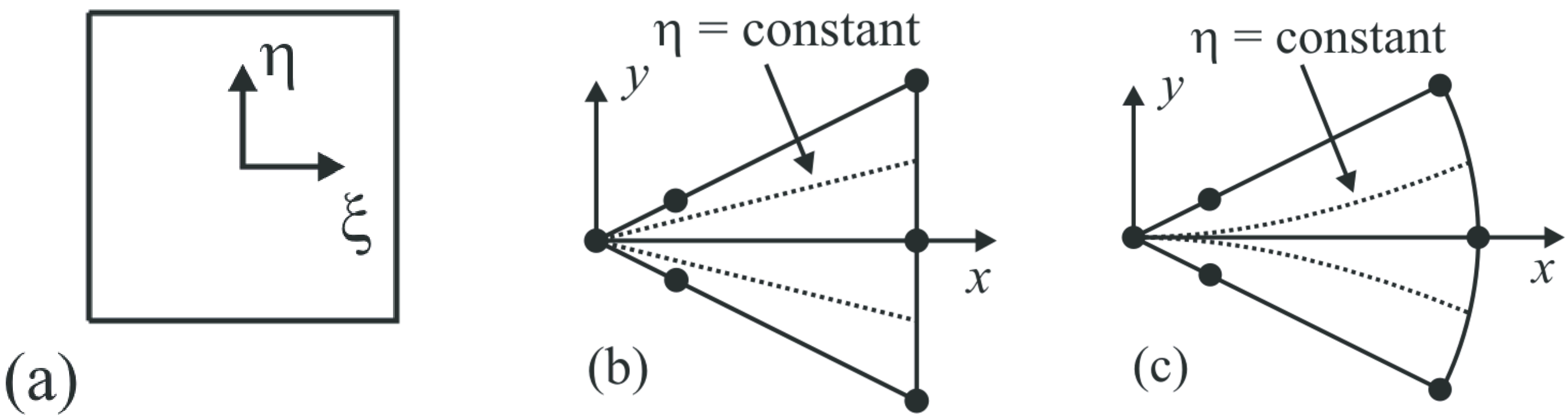
¼-point strain field: $$\varepsilon = \frac{d\mathrm{u}}{dr} = 2(\mathrm{u}_1 - 2\mathrm{u}_2 + \mathrm{u}_3)\frac{1}{l} + (-\tfrac{3}{2}\mathrm{u}_1 + 2\mathrm{u}_2 - \tfrac{1}{2}\mathrm{u}_3)\frac{1}{\sqrt{lr}}$$

Compare to theoretical field: $$\sigma_x = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^1 \left[\left(2 + \frac{n}{2} + (-1)^n \right) \cos(\frac{n}{2} - 1)\theta - (\frac{n}{2} - 1)\cos(\frac{n}{2} - 3)\theta \right]$$

Observations about this class of singular element

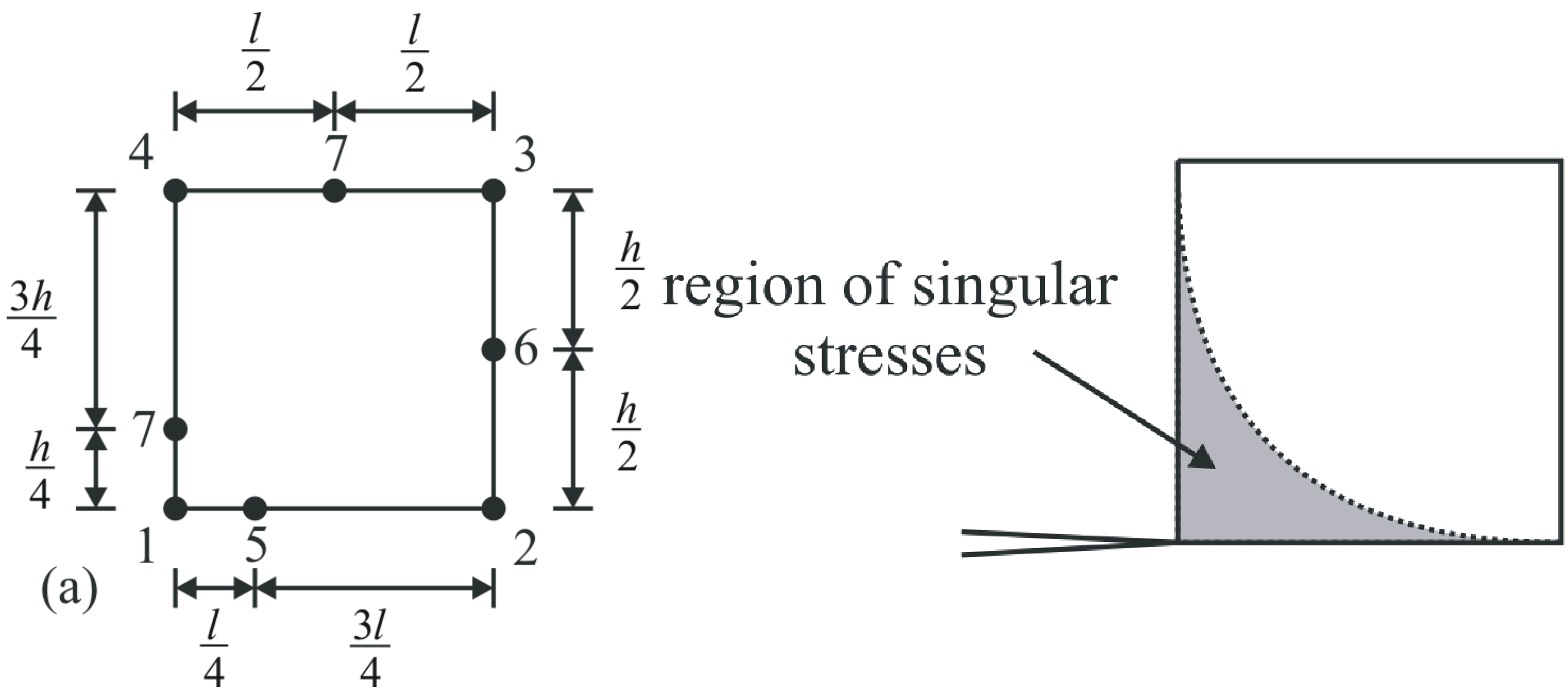
- The ¼-point quadratic element reproduces the correct leading displacement and strain terms.
- They support rigid body motion (constant strain), and linear strain (stress for LEFM).
- Higher order elements have analogous capabilities: eg. cubic elements reproduce the correct leading displacement and strain terms, support rigid body motion (constant strain), and linear strain (stress for LEFM), and include one more of the higher order theoretical terms, by locating the 2 side nodes at 1/9 and 4/9 points….
- No special software coding is needed, only perturbed nodal geometry. Therefore, any FEM/BEM code supporting quadratic or higher order elements support this class of singular element.
- There are subtleties about which native form to use, collapsed quadrilateral or natural triangle, and what happens when the element has curved sides. See Chapter 2 of Ingraffea and Wawrzynek for details.
- There are also some differences in completing the picture for BEM.

Bad Things Happen When Collapsed Quads are Distorted



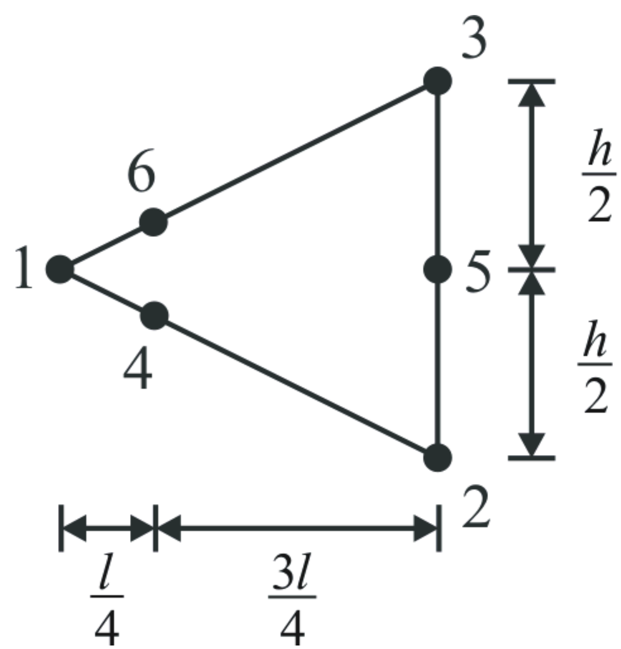
(a) the parametric space of a quadrilateral element, (b) and (c) the mapping of constant η lines in the Cartesian space for different far side node placements for collapsed quadrilateral quarter-point elements.

The ¼-Point Quad Is Also a Bad Deal



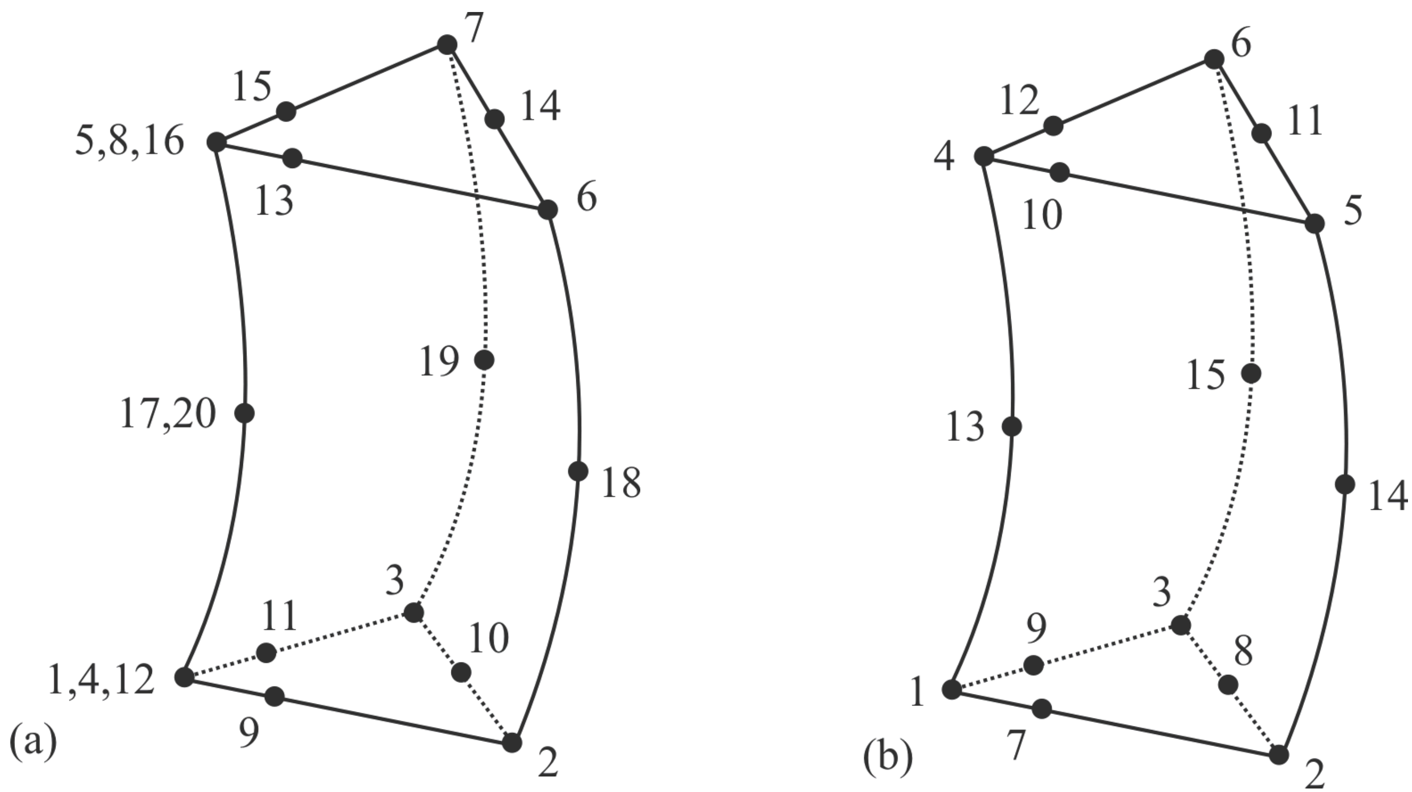
A schematic of the neighborhood where square root singularity is modeled in quadrilateral quarter-point elements.

Bad Things Do Not Happen When Natural Triangles, T6 elements, are Distorted



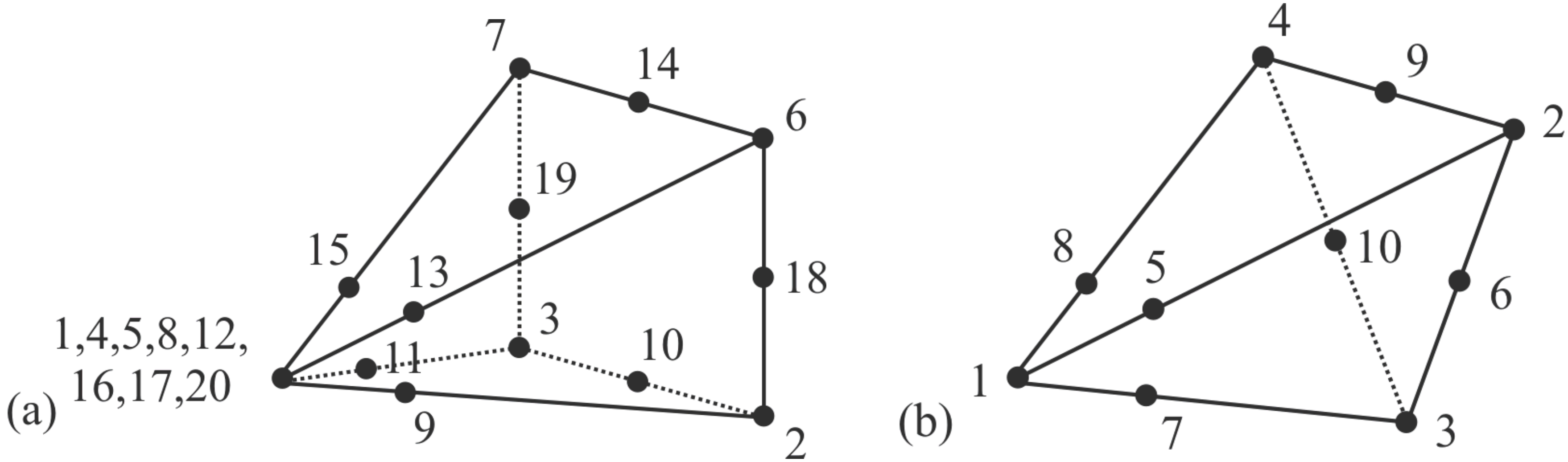
The natural triangle, T6, quarter-point element.

3D FEM, ¼-Point Singular Elements



The collapsed, 20-noded brick, quarter-point element (a), and (b), the natural 15-noded, wedge quarter-point element.

3D FEM, ¼-Point Singular Elements



The collapsed, 20-noded brick, pyramid quarter-point element (a), and (b), the natural, 10-noded tetrahedral, quarter-point element.

Observations about this class of singular element, 3D

- The ¼-point quadratic element reproduces the correct leading displacement and strain terms.
- They support rigid body motion (constant strain), and linear strain (stress for LEFM).
- Higher order elements have analogous capabilities: eg. cubic elements reproduce the correct leading displacement and strain terms, support rigid body motion (constant strain), and linear strain (stress for LEFM), and include one more of the higher order theoretical terms, by locating the 2 side nodes at 1/9 and 4/9 points….
- No special software coding is needed, only perturbed nodal geometry. Therefore, any FEM/BEM code supporting quadratic or higher order elements support this class of singular element.
- There are subtleties about which native form to use, collapsed brick or natural tetrahedron/wedge, and what happens when the element has curved faces. See Chapter 2 of Ingraffea and Wawrzynek for details.
- There are also some differences in completing the picture for BEM.

Questions We Have Yet to Answer for Which There Are Answers!

- How big should singular elements be?

- How many should be arrayed around a crack front?

- How best to extract SIF's from their computed fields?