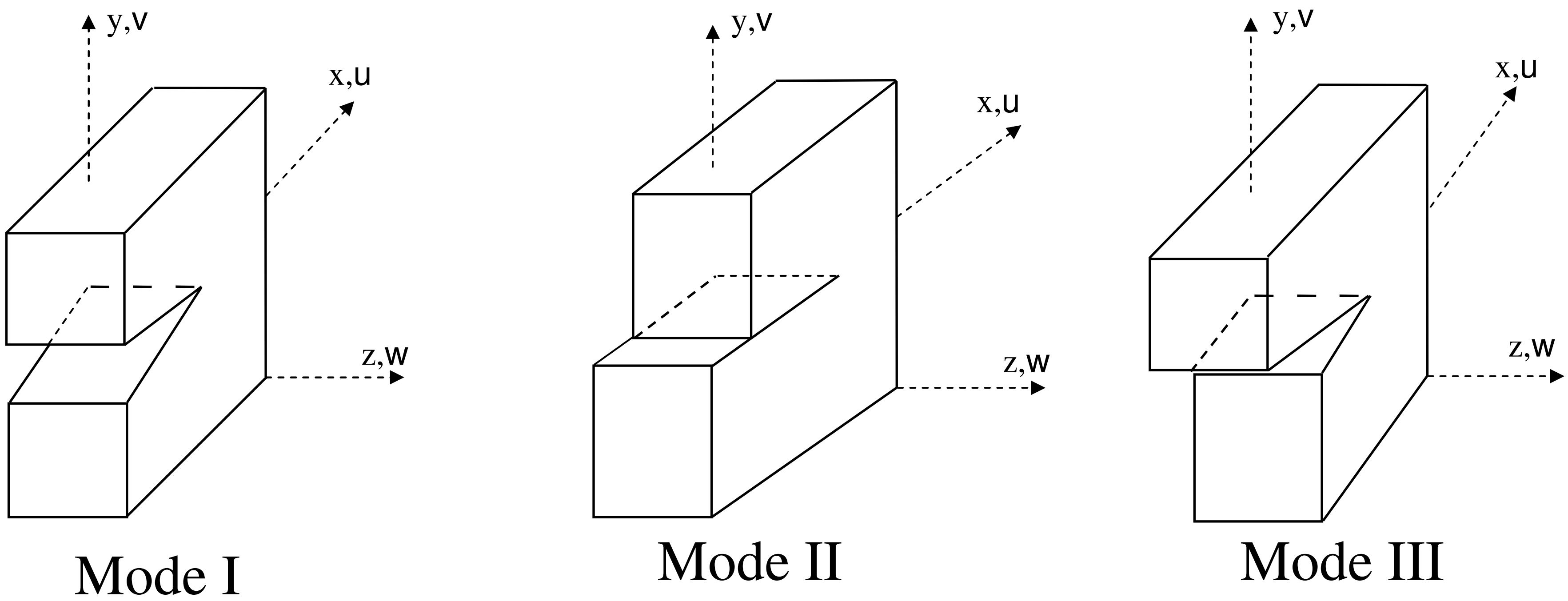
CEE 770 Meeting 2

Objectives of This Meeting

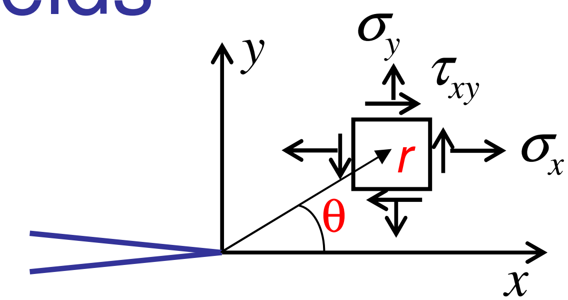
- Review some basic elements of LEFM to prepare for computational implementation:
  - ✓ Crack front stress and displacement fields
  - ✓ Stress Intensity Factor
  - ✓ T-stress
  - ✓ Small scale yielding (SSY) concept
  - ✓ Energy Release Rate
- Demonstrate finite element calculation of these fields using FRANC2D

Continuum Fracture Modes



Basic modes of crack loading. **Positive sense** shown for each:
Mode I = crack opening
Mode II = in-plane sliding
Mode III = anti-plane tearing

2D Crack Tip Fields



Williams **(1957) expansion of crack tip stress and displacement fields:**

Mode I

$$\sigma_x = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^I \left[\left(2 + \frac{n}{2} + (-1)^n \right) \cos(\frac{n}{2}-1)\theta - (\frac{n}{2}-1)\cos(\frac{n}{2}-3)\theta \right] \quad (1)$$

$$\sigma_y = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^I \left[\left(2 - \frac{n}{2} - (-1)^n \right) \cos(\frac{n}{2}-1)\theta + (\frac{n}{2}-1)\cos(\frac{n}{2}-3)\theta \right] \quad (2)$$

$$\tau_{xy} = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^I \left[(\frac{n}{2}-1)\sin(\frac{n}{2}-3)\theta - \left(\frac{n}{2} + (-1)^n \right) \sin(\frac{n}{2}-1)\theta \right] \quad (3)$$

$$\text{u} = \sum_{n=1}^{\infty} \frac{r^{\frac{n}{2}}}{2\mu} a_n^I \left[\left(\kappa + \frac{n}{2} + (-1)^n \right) \cos\frac{n}{2}\theta - \frac{n}{2}\cos(\frac{n}{2}-2)\theta \right] \quad (4)$$

$$\text{v} = \sum_{n=1}^{\infty} \frac{r^{\frac{n}{2}}}{2\mu} a_n^I \left[\left(\kappa - \frac{n}{2} - (-1)^n \right) \sin\frac{n}{2}\theta + \frac{n}{2}\sin(\frac{n}{2}-2)\theta \right] \quad (5)$$

$\sigma_z = 0$
plane stress

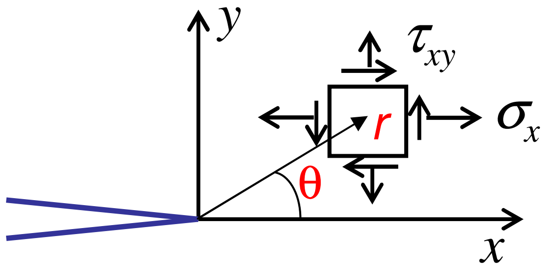
$\sigma_z = \nu\left(\sigma_x + \sigma_y\right)$
plane strain

$\tau_{xz} = \tau_{yz} = 0$

and where μ = G
and κ =
$3-4\nu$, plane stress
$(3-\nu)/(1+\nu)$, plane strain

Crack Tip Fields

Williams **expansion of crack tip stress and displacement fields (1957):**



Mode II

$$\sigma_x = -\sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^{II} \left[\left(2 + \frac{n}{2} - (-1)^n \right) \sin(\frac{n}{2}-1)\theta - (\frac{n}{2}-1)\sin(\frac{n}{2}-3)\theta \right] \quad (6)$$

$$\sigma_y = -\sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^{II} \left[\left(2 - \frac{n}{2} + (-1)^n \right) \sin(\frac{n}{2}-1)\theta + (\frac{n}{2}-1)\sin(\frac{n}{2}-3)\theta \right] \quad (7)$$

$$\tau_{xy} = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^{II} \left[(\frac{n}{2}-1)\cos(\frac{n}{2}-3)\theta - \left(\frac{n}{2} - (-1)^n \right) \cos(\frac{n}{2}-1)\theta \right] \quad (8)$$

$$\text{u} = -\sum_{n=1}^{\infty} \frac{r^{\frac{n}{2}}}{2\mu} a_n^{II} \left[\left(\kappa + \frac{n}{2} - (-1)^n \right) \sin\frac{n}{2}\theta - \frac{n}{2}\sin(\frac{n}{2}-2)\theta \right] \quad (9)$$

$$\text{v} = \sum_{n=1}^{\infty} \frac{r^{\frac{n}{2}}}{2\mu} a_n^{II} \left[\left(\kappa - \frac{n}{2} + (-1)^n \right) \cos\frac{n}{2}\theta + \frac{n}{2}\cos(\frac{n}{2}-2)\theta \right] \quad (10)$$

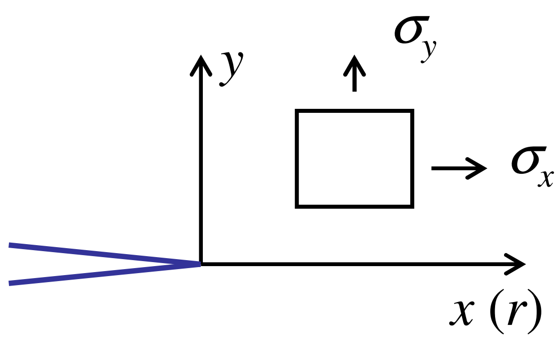
$\sigma_z = 0$
plane stress

$\sigma_z = \nu(\sigma_x + \sigma_y)$
plane strain

$\tau_{xz} = \tau_{yz} = 0$

where μ = G
and κ =
$3-4\nu$, plane stress
$(3-\nu)/(1+\nu)$, plane strain

Example of Expansion Along Crack Line, x = r, Mode I



$$\sigma_x = \frac{a_1}{\sqrt{r}} + 4a_2 + 3a_3\sqrt{r} + 8a_4 r + 5a_5 r^{3/2} + \cdots \qquad (11)$$

$$\sigma_y = \frac{a_1}{\sqrt{r}} + 3a_3\sqrt{r} + 5a_5 r^{3/2} + \cdots \qquad (12)$$

First (leading), or singular term, a_1: contains the *stress intensity factor*

Second term, a_2: contains the *T*-stress

Third term, a_3: the leading higher order term (note: non-polynomial!)

Definition of Stress Intensity Factor and T-stress from these Fields

Neglecting all but the first, singular term of this stress field results in the formal definition of the stress intensity factor:

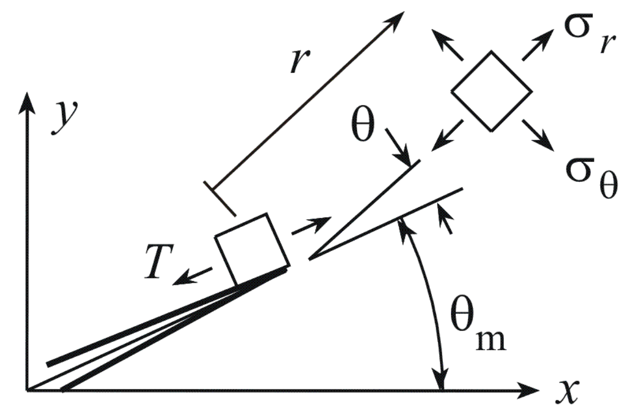
$$K_I = \lim_{r \to 0} \sigma_{yy} \sqrt{2 \pi r} \qquad (13)$$

$$K_{II} = \lim_{r \to 0} \tau_{xy} \sqrt{2 \pi r} \qquad (14)$$

$$K_{III} = \lim_{r \to 0} \tau_{yz} \sqrt{2 \pi r} \qquad (15)$$

The so-called T-stress is the constant stress acting parallel to the crack direction.

In Cylindrical Coordinates, to 2nd Term



$$\sigma_{\theta\theta} = \frac{1}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[K_I \cos^2\frac{\theta}{2} - \frac{3}{2}K_{II}\sin\theta\right] + \frac{T}{2}\left(1-\cos 2\theta\right) \qquad (16)$$

$$\sigma_{rr} = \frac{1}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[K_I\left(1+\sin^2\frac{\theta}{2}\right) + \frac{3}{2}K_{II}\sin\theta - 2K_{II}\tan\frac{\theta}{2}\right] + \frac{T}{2}\left(1+\cos 2\theta\right) \qquad (17)$$

$$\sigma_{r\theta} = \frac{1}{2\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[K_I \sin\theta + K_{II}\left(3\cos\theta - 1\right)\right] - \frac{T}{2}\sin 2\theta \qquad (18)$$

Crack Front Principal Stresses

$$\sigma_1 = \frac{K_1}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\right) \tag{19}$$

$$\sigma_2 = \frac{K_1}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\right) \tag{20}$$

$\sigma_3 = 0$ or $\sigma_3 = \dfrac{2\, v\, K_I}{\sqrt{2\pi\, r}} \cos\dfrac{\theta}{2}$ Plane stress, or plane strain

(21)

Mode III Fields, Plane Strain

$$\tau_{xz} = \frac{K_{III}}{(2\pi r)^{1/2}} \sin\frac{\theta}{2} \qquad (22) \qquad\qquad \tau_{yz} = \frac{K_{III}}{(2\pi r)^{1/2}} \cos\frac{\theta}{2} \qquad (23)$$

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xz} = 0$$

$$w = \frac{K_{III}}{G}\left[\frac{2r}{\pi}\right]^{1/2} \sin\frac{\theta}{2} \qquad (24)$$

$$\mathrm{u} = \mathrm{v} = 0$$

$$\text{for plane stress, let } \nu = \frac{\nu}{1+\nu}$$

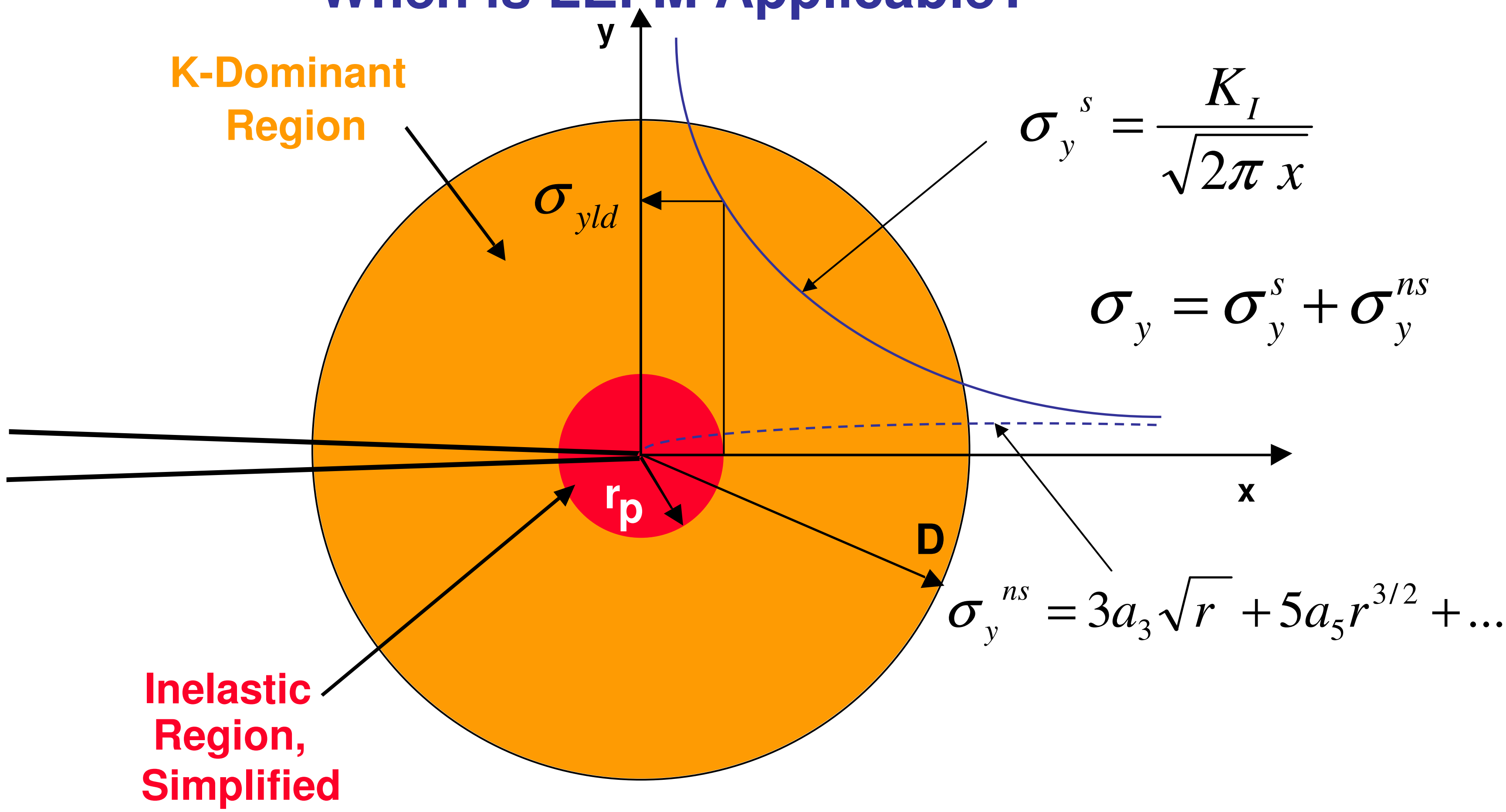
So Why is the Stress Intensity Factor so Important?

- Under conditions of small-scale yielding, all crack front fields are dominated (controlled) by the stress intensity factor.

- Therefore, all crack behavior:
  - Stability—will the crack tip move?
  - Trajectory– in what direction?
  - Rate– how fast?

  is controlled by the stress intensity factor and, maybe, the T-stress.

The Concept of K-Dominance: When is LEFM Applicable?



If r_p << D, K_I still controls fracture process.

Energy Release Rate

Recall that, in LEFM, energy release rate (crack driving *force*) is a dual of stress intensity. For example, in Mode I:

$$G_I = \frac{K_I^2}{E'} \qquad (25)$$

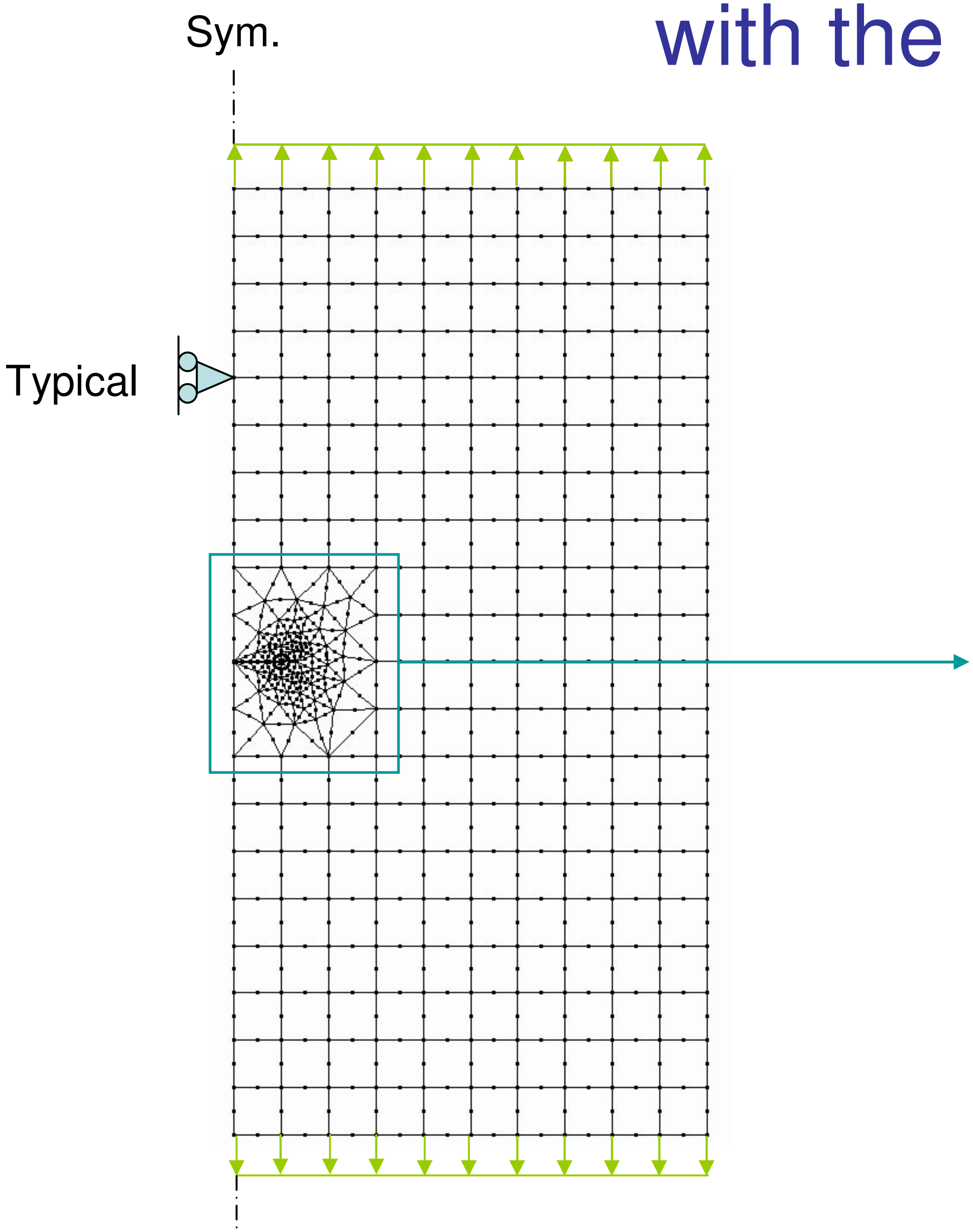
where

$$E' = E \quad for\ plane\ stress$$

$$E' = \frac{E}{(1-\nu^2)} \quad for\ plane\ strain$$

We will first concentrate on computing stress intensity factors, then, later, energy release rates (and their derivatives!).

How Well Can We Reproduce These Fields with the FEM?



The "Griffith Problem"